TOPOLOGY OF REAL ALGEBRAIC SETS OF DIMENSION 4: NECESSARY CONDITIONS

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ABSTRACT. Operators on the ring of algebraically constructible functions are used to compute local obstructions for a four-dimensional semialgebraic set to be homeomorphic to a real algebraic set. The link operator and arithmetic operators yield $2^{43} - 43$ independent characteristic numbers mod 2, which generalize the Akbulut-King numbers in dimension three.

The ring of algebraically constructible functions on a real algebraic set was introduced in [5]. The link operator on this ring was used to give a new description of the Akbulut-King numbers of three-dimensional stratified sets [2], as well as to generalize the topological conditions on algebraic sets discovered by Coste and Kurdyka [4].

Akbulut and Kurdyka have asked what invariants can be constructed by our method for four-dimensional sets. In this paper we produce a large number of independent new local topological conditions satisfied by algebraic sets of dimension four. Thus, in particular, there are four-dimensional semialgebraic sets which have vanishing Akbulut-King invariants, but which are not homeomorphic to algebraic sets. We do not know whether a four-dimensional semialgebraic set satisfying all of our conditions, as well as the Akbulut-King conditions, must be homeomorphic to an algebraic set.

The properties of the link operator Λ on constructible functions are reviewed in section 1. The main result of [5] (see also [6], [7]) is that the operator $\tilde{\Lambda} = \frac{1}{2}\Lambda$ preserves the set of algebraically constructible functions: If φ is algebraically constructible, so is $\tilde{\Lambda}\varphi$. Thus, in particular, if X is a real algebraic set with characteristic function $\mathbf{1}_X$, every function obtained from $\mathbf{1}_X$ using the arithmetic operations $+$, $-$, $*$, and the operator $\tilde{\Lambda}$ is integer-valued. Sets with this property we call *completely euler*. The property that $\tilde{\Lambda}\mathbf{1}_X$ is integer-valued is equivalent to Sullivan's condition [8] that X is *euler*: for all $x \in X$, the link of x in X has even euler characteristic.

In section 2 we show how to construct systematically invariants of a set X which vanish if and only if X is completely euler. These invariants are local, and if we assume that all the links of points of X are completely euler, then X is completely euler if and only if a finite list of mod 2 characteristic numbers vanish for each link. For X of dimension at most 4 we find $2^{29} - 29$ such characteristic numbers. Section 3 contains a proof of the independence of these numbers, by construction of examples which distinguish them.

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There are also arithmetic operators with rational coefficients which preserve the set of algebraically constructible functions. For example, if φ is algebraically constructible, then so is $P(\varphi) = \frac{1}{2}(\varphi^4 - \varphi^2)$. In section 4 we characterize such operators, and we show that P is the only such operator which gives new conditions on the topology of 4-dimensional algebraic sets. Using the operator P together with the operator $\tilde{\Lambda}$, we enhance our previous construction to produce $2^{43} - 43$ independent local characteristic numbers which vanish for algebraic sets of dimension ≤ 4.

We work with real semialgebraic sets in euclidean space, semialgebraic Whitney stratifications, and semialgebraically constructible functions. We use the foundational results that the link of a point in a semialgebraic set is well-defined up to semialgebraic homeomorphism, and that a semialgebraic set has a semialgebraic, locally semialgebraically trivial Whitney stratification (*cf.* [4]). Since semialgebraic sets are triangulable and our constructions are purely topological, we could just as well work with piecewise-linear sets, stratifications, and constructible functions.

It is not natural to assume that a real semialgebraic set has the same local dimension at every point, so we must be careful in dealing with dimension. By the dimension of a semialgebraic set X we mean its topological dimension, denoted $\dim X$. If T is a semialgebraic triangulation of X, then $\dim X$ is the maximum dimension of a simplex of T. For $x \in X$, the local dimension $\dim_x X$ of X at x is the maximum dimension of a closed simplex of T containing x. Thus $\dim X = \max\{\dim_x X, x \in X\}$, and if L is the link of x in X, then $\dim L = \dim_x X - 1$, which may be less than $\dim X - 1$.

1. Algebraically Constructible Functions

1.1. **Definition and main properties.** Let $X \subset \mathbb{R}^n$ be a real algebraic set. Following [5] we say that an integer-valued function $\varphi : X \to \mathbb{Z}$ is *algebraically constructible* if there exists a finite collection of algebraic sets Z_i and proper regular morphisms $f_i : Z_i \to X$ such that φ admits a presentation as a finite sum

$$\varphi(x) = \sum m_i \chi(f_i^{-1}(x)) \tag{1.1}$$

with integer coefficients m_i, where χ is the euler characteristic. We recall from [6], [7] that $\varphi : X \to \mathbb{Z}$ is algebraically constructible if and only if there exists a finite set of polynomials $g_1, \ldots, g_s \in \mathbb{R}[x_1, \ldots, x_n]$ such that

$$\varphi(x) = \operatorname{sgn} g_1(x) + \ldots + \operatorname{sgn} g_s(x). \tag{1.2}$$

The set of algebraically constructible functions on X forms a ring, which we denote by $\mathrm{A}(X)$. An algebraically constructible function is constructible in the usual sense; that is, it is a finite combination of characteristic functions of semialgebraic subsets X_i of X,

$$\varphi = \sum n_i \mathbf{1}_{X_i} \tag{1.3}$$

with integer coefficients n_i. The converse is not true; there exist constructible functions which are not algebraically constructible (see [5]).

Let X be a semialgebraic set, and let φ be a constructible function on X given by (1.3). Without loss of generality we may assume that all X_i are closed in X. If, moreover, all X_i

are compact then we define the *euler integral* of φ by

$$\int \varphi \, d\chi = \sum n_i \chi(X_i).$$

It is easy to see that $\int \varphi \, d\chi$ is well-defined; it depends only on φ and not on the presentation (1.3). The euler integral is uniquely determined by the properties that it is linear,

$$\int \varphi + \psi \, d\chi = \int \varphi \, d\chi + \int \psi \, d\chi,$$

and the euler integral of the characteristic function of a compact semialgebraic set is its euler characteristic,

$$\int \mathbf{1}_W \, d\chi = \chi(W).$$

If φ is given by (1.3), then for $x \in X$, the *link* of φ at x is defined by

$$\Lambda \, \varphi(x) = \sum n_i \chi(X_i \cap S(x, \varepsilon)),$$

where $S(x, \varepsilon)$ is a sufficiently small sphere centered at x. Thus

$$\Lambda \, \varphi(x) = \int_L \varphi \, d\chi,$$

where $L = X \cap S(x, \varepsilon)$ is the link of x in X. It is easy to see that $\Lambda \, \varphi$ is a well-defined constructible function on X. The link operator Λ is linear, and it satisfies

$$\Lambda \, \Lambda = 2 \, \Lambda \tag{1.4}$$

(see [5]). If φ has compact support then the euler integral of the link of φ is zero [5],

$$\int \Lambda \, \varphi \, d\chi = 0. \tag{1.5}$$

1.2. **Completely euler spaces.** Let $\tilde{\Lambda} = \frac{1}{2} \Lambda$. The following result is proved in [5]; see also [6].

Theorem 1.1. *If $\varphi \in \mathrm{A}(X)$, then $\Lambda \, \varphi$ is an even-valued function and $\tilde{\Lambda} \, \varphi \in \mathrm{A}(X)$.*

Suppose now that the semialgebraic set X is homeomorphic to an algebraic set. Then by Theorem 1.1 all functions constructed from $\mathbf{1}_X$ by means of the arithmetic operations $+, -, *$, and the operator $\tilde{\Lambda}$, are integer-valued. This observation was used in [5] to recover the Akbulut-King conditions [2] on the topology of real algebraic sets of dimension ≤ 3.

We say that the constructible function φ on X is *euler* if $\Lambda \, \varphi(x)$ is even for all $x \in X$, *i.e.* $\tilde{\Lambda} \, \varphi$ is integer-valued. The semialgebraic set X is *euler* if $\mathbf{1}_X$ is euler. We say that φ is *completely euler* if all functions obtained from φ by using the operations $+$, $-$, $*$, $\tilde{\Lambda}$ are integer-valued. The semialgebraic set X is *completely euler* if $\mathbf{1}_X$ is completely euler. Thus a necessary condition for a semialgebraic set X to be homeomorphic to an algebraic set is that X is completely euler.

The following theorem follows easily from the theory of basic open sets in real algebraic geometry [3]; for the proof see [5].

Theorem 1.2. *If X is an algebraic set of dimension at most d, then every constructible function on X divisible by 2^d is algebraically constructible.*

The following related observation will be useful in the sequel. If X is a semialgebraic set, let $\mathcal{I}(X)$ be the set of constructible functions φ on X such that, for all $k \geq 0$, $\dim(\operatorname{supp}\varphi \pmod{2^k}) < k$. Note that $\mathcal{I}(X)$ is an ideal of the ring of constructible functions on X.

Proposition 1.3. *$\tilde{\Lambda}\,\mathcal{I}(X) \subset \mathcal{I}(X)$.*

The proof is elementary (*cf.* [5], Prop. 3.1). It is easy to see that Theorem 1.2 implies that $\mathcal{I}(X) \subset \mathrm{A}(X)$, while Theorem 1.1 says that $\tilde{\Lambda}\,\mathrm{A}(X) \subset \mathrm{A}(X)$.

1.3. **Properties of $\tilde{\Lambda}$ and $\tilde{\Omega}$.** Define an operator Ω on constructible functions by $\Omega\,\varphi = 2\varphi - \Lambda\,\varphi$, and set $\tilde{\Omega} = \frac{1}{2}\,\Omega$. Thus

$$\tilde{\Lambda} + \tilde{\Omega} = \mathrm{I},$$

where I is the identity operator. The following elementary properties of $\tilde{\Lambda}$ and $\tilde{\Omega}$ will be used repeatedly:

(a) **Composition**:

$$\tilde{\Lambda}\,\tilde{\Lambda} = \tilde{\Lambda}, \quad \tilde{\Omega}\,\tilde{\Omega} = \tilde{\Omega}, \quad \tilde{\Lambda}\,\tilde{\Omega} = 0, \quad \tilde{\Omega}\,\tilde{\Lambda} = 0.$$

(b) **Integral**:

$$\int \tilde{\Lambda}\,\varphi\, d\chi = 0, \quad \int \tilde{\Omega}\,\varphi\, d\chi = \int \varphi\, d\chi.$$

(c) **Support**:

Let l be a nonnegative integer. If $\dim(\operatorname{supp}\varphi) \leq 2l$, then $\dim(\operatorname{supp}(\tilde{\Lambda}\,\varphi)) \leq 2l - 1$. If $\dim(\operatorname{supp}\varphi) \leq 2l + 1$, then $\dim(\operatorname{supp}(\tilde{\Omega}\,\varphi)) \leq 2l$.

(d) **Slice**:

Let $W \subset X$ be given by $W = H \cap X$, where X is a semialgebraic set in $\mathbb{R}^n$ and H is a smooth $(n-1)$-dimensional semialgebraic subset of $\mathbb{R}^n$ which is transverse to a semialgebraic Whitney stratification $\mathcal{S}$ of X. If φ is a constructible function on X which is constant on the strata of $\mathcal{S}$, then

$$(\tilde{\Lambda}\,\varphi)|W = \tilde{\Omega}(\varphi|W), \quad (\tilde{\Omega}\,\varphi)|W = \tilde{\Lambda}(\varphi|W).$$

Properties (a) and (b) follow from the corresponding properties of Λ ((1.4), (1.5)). Properties (c) and (d) follow from the definition of Λ by the following topological arguments.

Given a constructible function φ on X, let $\mathcal{S}$ be a locally trivial stratification of X such that φ is constant on the strata of $\mathcal{S}$. Let $k = \dim(\operatorname{supp}\varphi)$, and let $x \in S$, where S is a k-dimensional stratum of $\mathcal{S}$. Then

$$\tilde{\Lambda}\,\varphi(x) = \tfrac{1}{2}\int_{X \cap S(x,\epsilon)} \varphi\, d\chi = \tfrac{1}{2}\int_{S \cap S(x,\epsilon)} \varphi\, d\chi.$$

For $\epsilon > 0$ sufficiently small, $S \cap S(x,\epsilon)$ is semialgebraically homeomorphic to a $(k-1)$-sphere. Thus if k is even, $\tilde{\Lambda}\,\varphi(x) = 0$; and if k is odd, $\tilde{\Lambda}\,\varphi(x) = \varphi(x)$, so $\tilde{\Omega}\,\varphi(x) = 0$.

Since restriction and the operators $\tilde{\Lambda}$ and $\tilde{\Omega}$ are linear, by the definition of $\Lambda\,\varphi$ it suffices to prove (d) for $\varphi = \mathbf{1}_X$. For all $x \in W$, $X \cap S(x,\epsilon)$ is semialgebraically homeomorphic to the

suspension of $W \cap S(x,\epsilon)$, since a neighborhood of x in X is semialgebraically homeomorphic to the product of an interval with a neighborhood of x in W. Thus

$$\begin{aligned}\chi(X \cap S(x,\epsilon)) &= 2 - \chi(W \cap S(x,\epsilon)),\\ \Lambda\,\mathbf{1}_X(x) &= 2 \cdot \mathbf{1}_W(x) - \Lambda\,\mathbf{1}_W(x),\\ \Lambda\,\varphi(x) &= 2(\varphi|W)(x) - \Lambda(\varphi|W)(x),\end{aligned}$$

which implies (d).

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2. Construction of local invariants from the link operator

2.1. **A general construction.** Let $X \subset \mathbb{R}^n$ be a semialgebraic set. We describe an algorithm for producing a list of $\mathbb{Z}/2$-valued obstructions (finite in number if X has a finite locally trivial stratification) the vanishing of which are necessary and sufficient for X to be completely euler. Let $\mathcal{S}$ be a locally trivial stratification of X, and let $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_d = X$, $d = \dim X$, be the skeletons of $\mathcal{S}$. Let $\tilde\Lambda(X)$ be the ring of all functions obtained from $\mathbf{1}_X$ by the operations $+$, $-$, $*$, $\tilde\Lambda$. Every such function is constant on the strata of $\mathcal{S}$ and takes rational values. By definition the set X is completely euler if and only if all functions in $\tilde\Lambda(X)$ are integer-valued.

We define a sequence of subrings of $\tilde\Lambda(X)$,

$$\tilde\Lambda_0(X) \subset \tilde\Lambda_1(X) \subset \tilde\Lambda_2(X) \subset \cdots,$$

inductively as follows. Let $\tilde\Lambda_0(X)$ be the ring generated by $\mathbf{1}_X$. For $k \geq 0$ let $\tilde\Lambda_{k+1}(X)$ be the ring generated by $\tilde\Lambda_k(X)$ and $\{\tilde\Lambda\,\varphi \mid \varphi \in \tilde\Lambda_k(X)\}$. Then $\tilde\Lambda(X)$ is the direct limit of the rings $\tilde\Lambda_k(X)$. For each $k \geq 0$, if all functions in $\tilde\Lambda_k(X)$ are integer-valued, we define obstructions for all functions in $\tilde\Lambda_{k+1}(X)$ to be integer-valued, and we show that if all functions in $\tilde\Lambda_d(X)$ are integer-valued, then so are all functions in $\tilde\Lambda(X)$.

Recall that $\mathcal{I}(X)$ is the ideal of constructible functions φ on X such that, for all $k \geq 0$, $\dim(\operatorname{supp}\varphi \pmod{2^k}) < k$. Let $k \geq 0$ and suppose that $\tilde\Lambda_k(X)$ is integer-valued. Moreover, suppose we have finite subsets $G_0, \ldots, G_{k-1}$ of $\tilde\Lambda_k(X)$ such that for $j = 0, \ldots, k-1$, $G_0 \cup \cdots \cup G_j$ is an additive generating set for $\tilde\Lambda_j(X)/(\mathcal{I}(X) \cap \tilde\Lambda_j(X))$, and for all $\varphi \in G_j$, $\operatorname{supp}\varphi \subset X_{d-j}$. Then $\tilde\Lambda_k(X)/(\mathcal{I}(X) \cap \tilde\Lambda_k(X))$ is additively generated by $G_0, \ldots, G_{k-1}$, together with the products

$$\Phi = \begin{cases} \varphi \prod (\tilde\Lambda\,\varphi_i)^{n_i}, & d-(k-1) \text{ even},\\ \varphi \prod (\tilde\Omega\,\varphi_i)^{n_i}, & d-(k-1) \text{ odd},\end{cases} \tag{2.1}$$

where $\varphi \in G_0 \cup \cdots \cup G_{k-1}$, $\varphi_i \in G_{k-1}$, and $n_i > 0$ for some i. By the support properties of $\tilde\Lambda$ and $\tilde\Omega$ (section 1.3(c)), $\operatorname{supp}\Phi \subset X_{d-k}$, so by definition of $\mathcal{I}(X)$ the quotient $\tilde\Lambda_k(X)/(\mathcal{I}(X) \cap \tilde\Lambda_k(X))$ is additively generated by $G_0 \cup \cdots \cup G_{k-1} \cup G_k$, where G_k is a finite set of representatives of the equivalence classes modulo 2^{d-k} of the functions Φ.

Recall that $\tilde\Lambda(\mathcal{I}(X)) \subset \mathcal{I}(X)$. Thus if all functions in $\tilde\Lambda_k(X)$ are integer-valued, then all functions in $\tilde\Lambda_{k+1}(X)$ are integer-valued if and only if all the functions $\varphi \in G_k$ are euler. We call the conditions that these functions are euler the *depth* $k+1$ *euler conditions* for X. Since the construction of the generating sets G_k terminates at $k = d$ (for $k = d+1$, $\operatorname{supp}\Phi = \emptyset$),

we have that $\tilde\Lambda_d(X)/(\mathcal{I}(X)\cap\tilde\Lambda_d(X)) = \tilde\Lambda(X)/(\mathcal{I}(X)\cap\tilde\Lambda(X))$. Thus X is completely euler if and only if the depth k euler conditions hold for all $k = 1,\dots,d$.

2.2. **Localization of the invariants.** The condition that X is completely euler is local. The constructible function φ on X is euler if and only if $\int_L \varphi\, d\chi \equiv 0 \pmod 2$ for all links L in X, so each euler condition for X is equivalent to the vanishing of a set of mod 2 obstructions. (If X has a finite locally trivial stratification then the number of semialgebraic homeomorphism types of links in X is finite, so the total number of obstructions is finite.)

It is easier to analyze these obstructions if we restrict our attention to a single link L in X. This has the advantage of reducing the dimension of the space we consider. By the slice property of the link operator (section 1.3(d)), if the constructible function φ on X is euler, then $\varphi|L$ is euler and $\int_L \varphi\, d\chi$ is even. Thus the compact semialgebraic set L is the link of a point in a completely euler space X if and only if the following conditions are satisfied:

(A) L is completely euler, *i.e.* for all $\varphi \in \tilde\Lambda(L)$, φ is integer-valued.

(B) For all $\varphi \in \tilde\Lambda(L)$, $\int \varphi\, d\chi$ is even.

We have seen that (A) is equivalent to a finite list of euler conditions for L. In the above discussion X is replaced by L, and $d = \dim X$ is replaced by $d' = \dim L$ (with $d' \le d-1$ if L is a link in X). If we assume that L is completely euler, then condition (B) can be reduced to the vanishing of a finite set of mod 2 invariants. This follows from the facts that the ring $\tilde\Lambda(L)/(2\mathcal{I}(L)\cap\tilde\Lambda(L))$ is additively finitely generated, and that if $\varphi \in 2\mathcal{I}(L)$ then $\int \varphi\, d\chi$ is even. (Note that if L is a link in X then $2\mathcal{I}(L)$ is the restriction to L of the ideal $\mathcal{I}(X)$.) To produce a finite set of additive generators for this ring, note as above that if $k \ge 1$ and we have finite subsets $G_0,\dots,G_{k-1}$ of $\tilde\Lambda(L)$ such that for $j = 0,\dots,k-1$, $G_0\cup\dots\cup G_j$ is an additive generating set for $\tilde\Lambda_j(X)/(2\mathcal{I}(X)\cap\tilde\Lambda_j(X))$ and for all $\varphi \in G_j$ $\operatorname{supp}\varphi \subset L_{d'-j}$, then $\tilde\Lambda_k(X)/(2\mathcal{I}(X)\cap\tilde\Lambda_k(X))$ is additively generated by $G_0\cup\dots\cup G_{k-1}\cup G_k$, where G_k is a finite set of representatives of the equivalence classes modulo $2^{d'-k+1}$ of the functions Φ (2.1). We call the integrals $\int \psi\, d\chi \pmod 2$, $\varphi \in G_k$ the *depth k characteristic numbers* of L. If L is completely euler, then $\int \varphi\, d\chi$ is even for all $\varphi \in \tilde\Lambda(L)$ if and only if the depth k characteristic numbers of L vanish for all $k = 0,\dots,d'-1$.

2.3. **Invariants in dimension 4.** Now we work out the euler conditions and characteristic numbers for a link L in a semialgebraic set X of dimension at most 4.

(A) *L is completely euler.* The computation of the euler conditions for a semialgebraic set of dimension at most 3 was done in [5]. We review it here as an illustration of the above general algorithm. The ring $\tilde\Lambda_0(L)$ is generated by $\mathbf{1}_L$, and the depth 1 euler condition is that $\mathbf{1}_L$ is euler (Sullivan's condition). Assume this, and let $\varphi = \tilde\Omega\,\mathbf{1}_L \in \tilde\Lambda_1(L)$. Then $\tilde\Lambda_1(L)/(\mathcal{I}(L)\cap\tilde\Lambda_1(L))$ is additively generated by $\mathbf{1}_L$, φ, φ^2, φ^3, since $\operatorname{supp}\varphi \subset L_2$ and $\varphi^4 \equiv \varphi^2 \pmod 4$. So the depth 2 euler conditions are that φ, φ^2, φ^3 are euler. But φ is euler since $\tilde\Lambda\,\varphi = \tilde\Lambda\,\tilde\Omega\,\mathbf{1}_L = 0$, and φ^2 and φ^3 are also euler since φ^2 and φ^3 are congruent to φ mod 2. Thus the depth 2 euler conditions are trivial.

Let $\beta = \tilde\Lambda(\varphi^2)$ and $\gamma = \tilde\Lambda(\varphi^3)$. The ring $\tilde\Lambda_2(L)/(\mathcal{I}(L)\cap\tilde\Lambda_2(L))$ is additively generated by $\mathbf{1}_L$, φ, φ^2, φ^3, and the products $\psi_{abc} = \varphi^a\beta^b\gamma^c$ with $b+c>0$. Since $\operatorname{supp}(\psi_{abc}) \subset L_1$, we may write

$$\psi_{abc} = \alpha^a\beta^b\gamma^c$$

where $\alpha = \varphi|L_1$. Thus $\tilde{\Lambda}_2(L)/(\mathcal{I}(L)\cap\tilde{\Lambda}_2(L))$ is additively generated by $\mathbf{1}_L$, φ, φ^2, φ^3, together with representatives of the equivalence classes mod 2 of the functions ψ_{abc}, namely β, γ, $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\alpha\beta\gamma$. Now β and γ are euler, since $\tilde{\Lambda}\,\beta = \tilde{\Lambda}\,\tilde{\Lambda}(\varphi^2) = \tilde{\Lambda}(\varphi^2) = \beta$, and similarly $\tilde{\Lambda}\,\gamma = \gamma$. So the depth 3 euler conditions on L are that the four functions $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\alpha\beta\gamma$, are euler. (It was proved in [5] that these four functions are euler for a semialgebraic set of dimension ≤ 3 if and only if the Akbulut-King conditions hold.) We conclude that L is completely euler if and only if the functions

$$\mathbf{1}_L,\ \varphi\beta,\ \varphi\gamma,\ \beta\gamma,\ \varphi\beta\gamma$$

are euler, where $\varphi = \tilde{\Omega}\,\mathbf{1}_L$, $\beta = \tilde{\Lambda}(\varphi^2)$, and $\gamma = \tilde{\Lambda}(\varphi^3)$.

(B) *For all $\varphi \in \tilde{\Lambda}(L)$, $\int \varphi\, d\chi$ is even.* There is one depth 0 characteristic number, $\int \mathbf{1}_L\, d\chi$ (mod 2). (The vanishing of this characteristic number is Sullivan's condition $\chi(L) \equiv 0$ (mod 2) for a link in an algebraic set.) Again consider $\varphi = \tilde{\Omega}\,\mathbf{1}_L$. The ring $\tilde{\Lambda}_1(L)/(2\mathcal{I}(L)\cap\tilde{\Lambda}_1(L))$ is additively generated by $\mathbf{1}_L$, φ, φ^2, φ^3, since $\operatorname{supp}\varphi \subset L_2$ and $\varphi^4 \equiv \varphi^2 + 2(\varphi - \varphi^3)$ (mod 8). So the depth 1 characteristic numbers are the euler integrals of φ, φ^2, and φ^3 (mod 2). But $\int \varphi\, d\chi = \int \tilde{\Omega}\,\mathbf{1}_L\, d\chi = \int \mathbf{1}_L\, d\chi$, and φ^2, φ^3 have the same euler integrals as φ mod 2, since they are congruent to φ mod 2. So there are no new characteristic numbers of depth one.

Now consider $\beta = \tilde{\Lambda}(\varphi^2)$ and $\gamma = \tilde{\Lambda}(\varphi^3)$. The ring $\tilde{\Lambda}_2(L)/(2\mathcal{I}(L)\cap\tilde{\Lambda}_2(L))$ is generated additively by $\mathbf{1}_L$, φ, φ^2, φ^3, together with representatives of the equivalence classes mod 4 of the products $\psi_{abc} = \alpha^a\beta^b\gamma^c$ with $b + c > 0$. (Here $\alpha = \varphi|L_1$ as before.) So it suffices to consider the exponents a, b, c equal to $0, 1, 2, 3$. This still leaves us with $60 = 4^3 - 4$ functions. Note that $\beta \equiv \beta^2 \equiv \beta^3$ (mod 2), so instead of β, β^2, β^3 we use β, $\beta_2 = \beta^2 - \beta$, $\beta_3 = \beta^3 - \beta$ as a set of additive generators for the ring generated by β modulo 4. The advantage of this set of generators is that β_2 and β_3 are even-valued. We let $\beta_1 = \beta$. Similarly we define α_1, α_2, α_3, γ_1, γ_2, γ_3. Now all products $\psi'_{abc} = \alpha_a\beta_b\gamma_c$ with two even-valued factors are divisible by 4. Therefore the following list of functions completes our additive generating set:

$$(S_1)\qquad \beta,\ \gamma;$$

$$(S_2)\qquad \alpha\beta,\ \alpha\gamma,\ \beta\gamma, \alpha\beta\gamma;$$

$$(S_3)\qquad \beta_2,\ \gamma_2,\ \beta_3,\ \gamma_3;$$

$$(S_4)\qquad \begin{gathered} \alpha\beta_2,\ \alpha\gamma_2,\ \beta\alpha_2,\ \beta\gamma_2,\ \gamma\alpha_2,\ \gamma\beta_2,\\ \alpha\beta_3,\ \alpha\gamma_3,\ \beta\alpha_3,\ \beta\gamma_3,\ \gamma\alpha_3,\ \gamma\beta_3,\\ \alpha\beta\gamma_2,\ \alpha\gamma\beta_2,\ \beta\gamma\alpha_2,\\ \alpha\beta\gamma_3,\ \alpha\gamma\beta_3,\ \beta\gamma\alpha_3. \end{gathered}$$

The characteristic numbers of depth 2 are the euler integrals mod 2 of these functions. Now $\int \beta\, d\chi = \int \tilde{\Lambda}(\varphi^2)\, d\chi = 0$ and $\int \gamma\, d\chi = \int \tilde{\Lambda}(\varphi^3)\, d\chi = 0$, so the integrals of the functions in S_1 are zero. The functions in S_3 or S_4 are even, and hence have even euler integrals. The

nontrivial characteristic numbers of depth 2 are the integrals of the functions in S_2 (mod 2):

$$\int \alpha\beta\, d\chi,\ \int \alpha\gamma\, d\chi,\ \int \beta\gamma\, d\chi,\ \int \alpha\beta\gamma\, d\chi \tag{2.2}$$

The vanishing of these 4 characteristic numbers is a natural generalization of the Akbulut-King conditions.

Let $S = S_1 \cup S_2 \cup S_3 \cup S_4$, and let $S' = S_2 \cup S_3 \cup S_4$. The ring $\tilde\Lambda_3(L)/(2\mathcal{I}(L) \cap \tilde\Lambda_3(L))$ is generated additively by $\mathbf{1}_L$, φ, φ^2, φ^3, and S, together with the equivalence classes mod 2 of the functions

$$\Phi(\mathbf{m},\mathbf{n}) = \alpha^{m_\alpha}\beta^{m_\beta}\gamma^{m_\gamma} \prod_{\psi\in S'} (\tilde\Omega(\psi))^{n_\psi}, \tag{2.3}$$

where $\mathbf{m} = (m_\alpha, m_\beta, m_\gamma)$, and $\mathbf{n} = (n_\psi)_{\psi\in S'}$, with m_α, m_β, m_γ, n_ψ equal to 0 or 1, and $\sum_{\psi\in S'} n_\psi > 0$. The support of $\Phi(\mathbf{m},\mathbf{n})$ is contained in L_0. Since L_0 is finite,

$$\int \Phi(\mathbf{m},\mathbf{n})\, d\chi = \sum_{p\in L_0} \Phi(\mathbf{m},\mathbf{n})(p). \tag{2.4}$$

Some of these integrals are automatically even. For instance,

$$\int \tilde\Omega\, \beta_2\, d\chi = \int \beta_2\, d\chi \equiv 0 \pmod 2,$$

since β_2 is even-valued. The same argument works for any other function from S_3 or S_4. On the other hand, for $\psi \in S_2$ we recover the depth 2 characteristic numbers $\int \tilde\Omega\, \psi\, d\chi = \int \psi\, d\chi$ (2.2). This leaves us with $2^3(2^{26} - 1) - 26 = 2^{29} - 34$ new characteristic numbers (2.4) of depth 3. We summarize our analysis of the link L as follows.

Theorem 2.1. *Let L be a compact semialgebraic set of dimension at most 3. Let $\varphi = \tilde\Omega\, \mathbf{1}_L$, $\beta = \tilde\Lambda\, \varphi^2$, $\gamma = \tilde\Lambda\, \varphi^3$. Then L is a link in a completely euler space if and only if the following conditions hold:*

1. *L is euler and $\varphi\beta$, $\varphi\gamma$, $\beta\gamma$, $\varphi\beta\gamma$ are euler (L satisfies the Akbulut-King conditions);*
2. *L has even euler characteristic and $\varphi\beta$, $\varphi\gamma$, $\beta\gamma$, $\varphi\beta\gamma$ have even euler integral;*
3. *The characteristic numbers*

$$\mathbf{a}(\mathbf{m},\mathbf{n}) = \sum_{p\in L_0} \Phi(\mathbf{m},\mathbf{n})(p),$$

defined above (2.4) are even.

The completely euler set L of dimension at most 3 has $2^{29} - 29$ characteristic numbers in all: the euler characteristic of L and the $2^{29} - 30$ characteristic numbers (2.4), which include the 4 numbers (2.2).

3. Independence of invariants. Construction of examples

In this section we show that the $2^{29} - 29$ characteristic numbers of Theorem 2.1 are independent, *i.e.* for any subset $\mathcal{C}$ of these characteristic numbers there exists a compact completely euler semialgebraic set L of dimension 3 such that exactly those characteristic numbers of L which are in $\mathcal{C}$ are nonzero (mod 2). Since the characteristic numbers are additive with

respect to disjoint union, it suffices to construct for each characteristic number a set L with exactly that characteristic number nonzero.

3.1. **Outline of the construction.** The set $L = S^3 \vee S^3$ (two 3-spheres identified at at point p) has the characteristic number $\chi(L) \equiv 1 \pmod 2$, and $\varphi = \tilde{\Omega}\, \mathbf{1}_L = \mathbf{1}_{\{p\}}$. Thus $\beta = \tilde{\Lambda}\, \varphi^2 = 0$ and $\gamma = \tilde{\Lambda}\, \varphi^3 = 0$, so L is completely euler and all the other characteristic numbers of L are zero.

So it remains to construct, for each characteristic number (2.4), a compact semialgebraic set L of dimension 3 such that L has even euler characteristic, L is euler, the functions $\varphi\beta$, $\varphi\gamma$, $\beta\gamma$, $\varphi\beta\gamma$ are euler, and exactly the given characteristic number of L is nonzero.

Such a set L will be constructed together with its filtration by skeletons of a stratification

$$L_0 \subset L_1 \subset L_2 \subset L_3 = L$$

in the following three steps. We make no attempt to minimize the number of strata in the construction of L.

1. Given a characteristic number $\mathbf{a}(\mathbf{m}, \mathbf{n})$, we construct L_1 and constructible functions α, β, γ on L_1 such that:
   (a) $\tilde{\Omega}\, \alpha = \tilde{\Omega}\, \beta = \tilde{\Omega}\, \gamma = 0$ and $\alpha\beta, \alpha\gamma, \beta\gamma$, $\alpha\beta\gamma$, and $\mathbf{1}_{L_1}$ are euler,
   (b) $\sum_{p \in L_0} \Phi(\mathbf{m}, \mathbf{n})(p)$ is the only sum as in (2.4) which is nonzero (mod 2).
   The conditions that $\tilde{\Omega}\, \alpha = 0$ and $\mathbf{1}_{L_1}$ is euler are not necessary, but they will help to simplify the construction of L.
2. Given $(L_1, \alpha, \beta, \gamma)$ satisfying 1(a), we construct L_2 and a constructible function φ on L_2 such that:
   (a) $\tilde{\Lambda}\, \varphi = 0$,
   (b) $\alpha \equiv \varphi|_{L_1}$, $\beta \equiv \tilde{\Lambda}\, \varphi^2$, $\gamma \equiv \tilde{\Lambda}\, \varphi^3 \pmod 4$ on open 1-dimensional strata, and the same congruences hold (mod 2) on L_0 ,
   (c) $\int \varphi \, d\chi$ is even .
3. Given (L_2, φ) satisfying 2(a), we construct L such that:
   (a) L is euler,
   (b) $\varphi \equiv \tilde{\Omega}\, \mathbf{1}_L \pmod 8$ on open 2-dimensional strata, $\varphi \equiv \tilde{\Omega}\, \mathbf{1}_L \pmod 4$ on open 1-dimensional strata, and $\varphi \equiv \tilde{\Omega}\, \mathbf{1}_L \pmod 2$ on L_0.
   Then, in particular, if L_2 also satisfies 2(c) then $\chi(L)$ is even: $\int \mathbf{1}_L \, d\chi = \int \tilde{\Omega}\, \mathbf{1}_L \, d\chi \equiv \int \varphi \, d\chi \pmod 2$.

Note that if $\psi \equiv \psi' \pmod{2^k}$, then $\tilde{\Lambda}\, \psi \equiv \tilde{\Lambda}\, \psi' \pmod{2^{k-1}}$. This implies that the characteristic number $\mathbf{a}(\mathbf{m}, \mathbf{n})$ depends only on the congruence classes of $\mathbf{1}_L \pmod{16}$, $\varphi \pmod 8$, or α, β, $\gamma \pmod 4$. We may also conclude the sightly stronger natural condition that $\mathbf{a}(\mathbf{m}, \mathbf{n})$ depends only on the congruence classes of $\mathbf{1}_L$, φ, α, β, γ modulo the ideal $2\mathcal{I}(L)$. Indeed, the conditions 2(b) and 3(b) above are modulo $2\mathcal{I}(L)$.

3.2. **Step 1. Construction of** $(L_1, \alpha, \beta, \gamma)$**.** As noted above, we will construct $(L_1, \alpha, \beta, \gamma)$ with the extra condition $\tilde{\Omega}\, \alpha = 0$. This will allow us to treat α, β, γ in the same way. All the characteristic numbers are defined modulo 2; on the other hand, the integer-valued functions α, β, γ on L_1 matter modulo 4.

3.2.1. *Elementary blocks, an example.* First, to illustrate the method, we consider only one function, say β. In this case there are 4 characteristic numbers:

$$\begin{aligned} &\sum_{p\in L_0} \beta(p)\,\tilde\Omega\,\beta_2(p), \qquad \sum_{p\in L_0} \beta(p)\,\tilde\Omega\,\beta_3(p), \\ &\sum_{p\in L_0} \tilde\Omega\,\beta_2(p)\,\tilde\Omega\,\beta_3(p), \qquad \sum_{p\in L_0} \beta(p)\,\tilde\Omega\,\beta_2(p)\,\tilde\Omega\,\beta_3(p), \end{aligned} \tag{3.1}$$

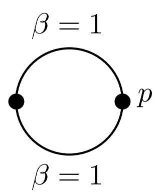
where $\beta_2 = \beta^2 - \beta$, $\beta_3 = \beta^3 - \beta \pmod 2$.

Let (L_1, β) be such that $\tilde\Omega\,\beta = 0$. The characteristic numbers (3.1) can be computed as follows. Let $p \in L_1$ be a vertex and suppose p is in the boundary of exactly k one-dimensional strata $S_1, \dots, S_k$, with the values of β on these strata equal to $l_1, \dots, l_k$, respectively. Then, since $\tilde\Omega\,\beta(p) = \beta(p) - \frac12 \sum l_i = 0$, we have

$$\begin{aligned} &\beta(p) = \tfrac12 \sum l_i, \\ &\tilde\Omega\,\beta_2(p) = \tilde\Omega\,\beta^2(p) = \beta^2(p) - \tfrac12 \sum l_i^2, \\ &\tilde\Omega\,\beta_3(p) = \tilde\Omega\,\beta^3(p) = \beta^3(p) - \tfrac12 \sum l_i^3. \end{aligned} \tag{3.2}$$

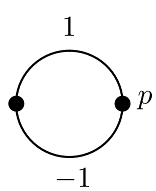
Now consider the following 1-dimensional stratified sets (with distinguished vertex p), which we call *elementary blocks*:

(A)



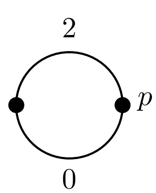
$(\beta, \tilde\Omega\,\beta_2, \tilde\Omega\,\beta_3)(p) = (1, 0, 0)$

(B)



$(\beta, \tilde\Omega\,\beta_2, \tilde\Omega\,\beta_3)(p) = (0, 1, 0)$

(C)



$(\beta, \tilde\Omega\,\beta_2, \tilde\Omega\,\beta_3)(p) = (1, 1, 1)$

For each block we indicate the values of β on the 1-dimensional strata. The values of β, $\tilde\Omega\,\beta_2$, $\tilde\Omega\,\beta_3$, given by (3.2), are indicated to the right of each diagram.

3.2.2. *Additivity.* The numbers in (3.1) are additive with respect to taking the wedge at the vertex p. More precisely, suppose we have two 1-dimensional stratified sets (L', β') and (L'', β''), with β' and β'' constructible functions on L' and L'' respectively, $\tilde\Omega\,\beta' = 0$, and

$\tilde{\Omega}\,\beta'' = 0$. Suppose we have chosen vertices $p' \in L'$ and $p'' \in L''$ as base points, and let (L, p) be the wedge $(L', p') \vee (L'', p'')$, with $p \in L$ the induced base point on L. Then $\beta'|_{L'\setminus p'}$ and $\beta''|_{L''\setminus p''}$ extend uniquely to a constructible function β on L satisfying $\tilde{\Omega}\,\beta = 0$, and

$$\begin{aligned} &\beta(p) = \beta'(p') + \beta''(p'') \\ &\tilde{\Omega}\,\beta_2(p) \equiv \tilde{\Omega}\,\beta_2'(p') + \tilde{\Omega}\,\beta_2''(p'') \pmod 2 \\ &\tilde{\Omega}\,\beta_3(p) \equiv \tilde{\Omega}\,\beta_3'(p') + \tilde{\Omega}\,\beta_3''(p'') \pmod 2. \end{aligned} \tag{3.3}$$

By wedging the elementary blocks (A), (B), (C), one may easily construct (L_1, β) with $\tilde{\Omega}\,\beta = 0$, $\mathbf{1}_{L_1}$ euler, and with precisely one of the numbers in (3.1) nonzero. For instance $(A) \vee (B)$ has only the first characteristic number $\sum_{p \in L_0} \beta(p)\,\tilde{\Omega}\,\beta_2(p)$ nonzero. We leave to the reader the construction of the remaining three examples.

3.2.3. *Elementary blocks, the general case.* We will define below an elementary block corresponding to each function of the set $S = S_1 \cup S_2 \cup S_3 \cup S_4$. For reasons of symmetry, instead of S_1, S_3, we consider the families

$$\alpha, \beta, \gamma; \tag{S_1'}$$

$$\alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3; \tag{S_3'}$$

We will use the elementary blocks to construct $(L_1, \alpha, \beta, \gamma)$ and a vertex $p \in L_1$ with given values of the following 31 expressions:

$$\alpha(p), \beta(p), \gamma(p); \tag{V_1}$$

$$\tilde{\Omega}(\alpha\beta)(p), \tilde{\Omega}(\alpha\gamma)(p), \tilde{\Omega}(\beta\gamma)(p), \tilde{\Omega}(\alpha\beta\gamma)(p); \tag{V_2}$$

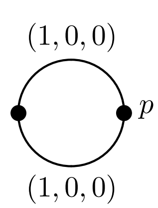
$$\tilde{\Omega}\,\alpha_2(p), \tilde{\Omega}\,\beta_2(p), \tilde{\Omega}\,\gamma_2(p), \tilde{\Omega}\,\alpha_3(p), \tilde{\Omega}\,\beta_3(p), \tilde{\Omega}\,\gamma_3(p); \tag{V_3}$$

$$\begin{gathered} \tilde{\Omega}(\alpha\beta_2)(p), \tilde{\Omega}(\alpha\gamma_2)(p), \tilde{\Omega}(\beta\alpha_2)(p), \tilde{\Omega}(\beta\gamma_2)(p), \tilde{\Omega}(\gamma\alpha_2)(p), \tilde{\Omega}(\gamma\beta_2)(p), \\ \tilde{\Omega}(\alpha\beta_3)(p), \tilde{\Omega}(\alpha\gamma_3)(p), \tilde{\Omega}(\beta\alpha_3)(p), \tilde{\Omega}(\beta\gamma_3)(p), \tilde{\Omega}(\gamma\alpha_3)(p), \tilde{\Omega}(\gamma\beta_3)(p), \\ \tilde{\Omega}(\alpha\beta\gamma_2)(p), \tilde{\Omega}(\alpha\gamma\beta_2)(p), \tilde{\Omega}(\beta\gamma\alpha_2)(p), \\ \tilde{\Omega}(\alpha\beta\gamma_3)(p), \tilde{\Omega}(\alpha\gamma\beta_3)(p), \tilde{\Omega}(\beta\gamma\alpha_3)(p). \end{gathered} \tag{V_4}$$

We will use the linear ordering of the set $V = V_1 \cup V_2 \cup V_3 \cup V_4$ given by the above list. For each expression v in V we will define an elementary block such that v is nonzero at all vertices of the block, and all expressions which follow v in the given ordering of V vanish at all vertices of the block. (The reader may easily check this property of the ordering of V.)

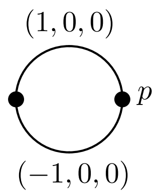
The elementary blocks are defined by simple pictures. On each picture we indicate the values of (α, β, γ) on the 1-dimensional strata. We illustrate 9 cases; the remaining 22 elementary blocks are defined by symmetry among the functions α, β, and γ.

(α)



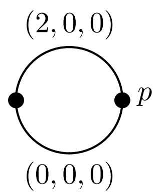
$\alpha(p) \neq 0$

(α_2)



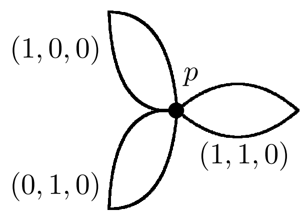
$\tilde{\Omega}\,\alpha_2(p) \neq 0$

(α_3)



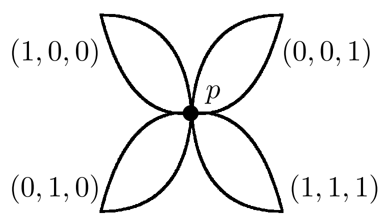
$\tilde{\Omega}\,\alpha_3(p) \neq 0$

$(\alpha\beta)$



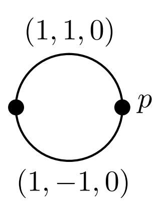
$\tilde{\Omega}(\alpha\beta)(p) \neq 0$

$(\alpha\beta\gamma)$



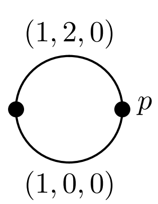
$\tilde{\Omega}(\alpha\beta\gamma)(p) \neq 0$

$(\alpha\beta_2)$



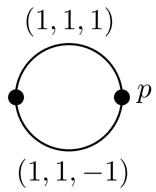
$\tilde{\Omega}(\alpha\beta_2)(p) \neq 0$

$(\alpha\beta_3)$



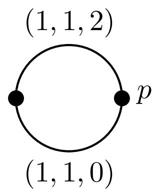
$\tilde{\Omega}(\alpha\beta_3)(p) \neq 0$

$(\alpha\beta\gamma_2)$



$\tilde{\Omega}(\alpha\beta\gamma_2)(p) \neq 0$

$(\alpha\beta\gamma_3)$



$\tilde{\Omega}(\alpha\beta\gamma)(p) \neq 0$

3.2.4. *Additivity, the general case.* We want the expressions in V to be additive with respect to taking the wedge of elementary blocks. This always holds for 1-dimensional stratified sets and the expressions V_1, V_3, V_4, but it can fail for V_2. For the expressions in V_1, V_3, V_4, additivity can be checked directly. (For $\beta(p)$ additivity follows from (3.2), and similarly $\alpha(p)$ and $\gamma(p)$ are additive. The expressions in $V_3 \cup V_4$ are of the form $\tilde{\Omega}(\psi)(p)$, where ψ is even, so $\tilde{\Omega}(\psi)(p) \equiv \tilde{\Lambda}(\psi)(p) \pmod 2$, and the latter is clearly additive.) However, consider the expression $\tilde{\Omega}(\alpha\beta)$ in V_2, for instance. The wedge $(L,p) = (L',p') \vee (L'',p'')$ of two 1-dimensional stratified sets $(L',\alpha',\beta',\gamma')$ and $(L'',\alpha'',\beta'',\gamma'')$ gives

$$\begin{aligned}\tilde{\Omega}(\alpha\beta)(p) =& (\alpha\beta)(p) - \tilde{\Lambda}(\alpha\beta)(p)\\ =&\alpha'\beta'(p') + \alpha''\beta''(p'') + (\alpha'(p')\cdot\beta''(p'') + \alpha''(p'')\cdot\beta'(p'))\\ &- (\tilde{\Lambda}(\alpha'\beta')(p') + \tilde{\Lambda}(\alpha''\beta'')(p''))\\ =&\tilde{\Omega}(\alpha'\beta')(p') + \tilde{\Omega}(\alpha''\beta'')(p'') + (\alpha'(p')\cdot\beta''(p'') + \alpha''(p'')\cdot\beta'(p')),\end{aligned}$$

since $\tilde{\Lambda}(\alpha\beta)$ is clearly additive. There are similar formulas for $\tilde{\Omega}(\alpha\gamma)$, $\tilde{\Omega}(\beta\gamma)$, and $\tilde{\Omega}(\alpha\beta\gamma)$.

Thus, even though the expressions in V_2 are not additive with respect to taking the wedge, we still have that additivity holds for them if one of the wedge factors is one of the elementary blocks $(\alpha\beta)$, $(\beta\gamma)$, $(\alpha\gamma)$, $(\alpha\beta\gamma)$, since $\alpha = \beta = \gamma = 0$ at the vertex of each of these blocks.

3.2.5. *Wedging elementary blocks.*

Proposition 3.1. *Given a subset U of V, there exists a compact 1-dimensional stratified set $(L_1,\alpha,\beta,\gamma)$ with a distinguished vertex $p_0 \in L_1$ such that*

1. $\tilde{\Omega}\,\alpha = \tilde{\Omega}\,\beta = \tilde{\Omega}\,\gamma = 0$, *and $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\alpha\beta\gamma$, and $\mathbf{1}_{L_1}$ are euler;*
2. $v(p_0) \neq 0$ *if and only if $v \in U$;*
3. *At every vertex $p \in L_1$, $p \neq p_0$, only one of the expressions in V is non-zero, and it is not an expression in V_2.*

Remark 3.2. Note that the sum over all $p \in L_1$ of an expression in V_1, V_3, or V_4 must be even. (Since all functions of S_3 and S_4 are even, for each such a function ψ we have that $\sum_{p\in L_0}\tilde{\Omega}\,\psi(p) = \int \tilde{\Omega}\,\psi\,d\chi = \int \psi\,d\chi$ is even. Similarly the sums $\sum_{p\in L_0}\alpha(p)$, $\sum_{p\in L_0}\beta(p)$,

$\sum_{p\in L_0}\gamma(p)$ are even.) This shows that the presence of nonzero expressions at vertices other than p_0 is necessary in the statement of the proposition.

Proof of Proposition 3.1. We construct L_1 by wedging elementary blocks. By definition of the elementary blocks, condition (1) is satisfied for any set constructed in this way.

The construction is by induction on the last element u of U with respect to the given ordering of V. To fix ideas suppose that u is in V_3 or V_4. (The cases $u \in V_1$ or $u \in V_2$ are much simpler; they are left to the reader.) Let (u) be the elementary block corresponding to u. The underlying set B_u of this block has two vertices p_1 and p_2, and u is nonzero at both vertices. It may happen that u is the only nonzero expression in V at the vertices of B_u. But in general there are some expressions in V, necessarily prior to u in the ordering of V, which are nonzero at both vertices of B_u. Denote the set of such expressions by V_u. By the inductive assumption applied to V_u, there exists $(L',\alpha',\beta',\gamma')$ satisfying conditions (2) and (3) for V_u, with distinguished vertex p_0'. Now wedge (B_u,p_1) and (L',p_0') to obtain a set with all the expressions of V_u vanishing at the vertex $p_0=p_1\vee p_0'$, except perhaps for some expressions in V_2. But any expression $v\in V_2$ which does not vanish at this vertex can be corrected by wedging with the correxponding elementary block (v). Thus we can ensure that all expressions in V except u vanish at p_0. The same procedure can be applied to the vertex p_2 of (u). In this way we obtain a set $(L_u,\alpha_u,\beta_u,\gamma_u)$ with distinguished vertex p_0 satisfying conditions (2) and (3) for the subset $\{u\}$ of V.

Now let $U'=U\setminus\{u\}$. If U' is empty we are done. Otherwise we apply the inductive assumption to U' and wedge the set $(L',\alpha',\beta',\gamma')$ we obtain with the set $(L_u,\alpha_u,\beta_u,\gamma_u)$. In this way we can ensure that the nonzero expressions at the distinguished vertex p_0 are exactly those of $\{u\}\cup U'=U$. More precisely, we must again correct the expressions $v\in V_2$ which do not vanish at p_0, by wedging with the corresponding blocks (v). □

Corollary 3.3. *Given a characteristic number* $\mathbf{a}(\mathbf{m},\mathbf{n})$*, there exists a compact stratified set* L_1 *of dimension 1 and constructible functions* α, β, γ *on* L_1 *such that*

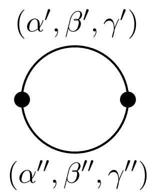
1. $\tilde\Omega\,\alpha=\tilde\Omega\,\beta=\tilde\Omega\,\gamma=0$ *and* $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\alpha\beta\gamma$, *and* $\mathbf{1}_{L_1}$ *are euler;*
2. $\sum_{p\in L_0}\Phi(\mathbf{m},\mathbf{n})(p)$ *is the only nonzero sum of type (2.4).*

Proof. Recall that $\Phi(\mathbf{m},\mathbf{n})=\alpha^{m_\alpha}\beta^{m_\beta}\gamma^{m_\gamma}\prod_{\psi\in S'}(\tilde\Omega(\psi))^{n_\psi}$, where the exponents are 0 or 1 and $\sum_{\psi\in S'}n_\psi>0$ (2.3). Let U be the subset of V consisting of those expressions which correspond to nontrivial factors of $\Phi(\mathbf{m},\mathbf{n})$. If we apply Proposition 3.1 to U, we obtain $(L_1,\alpha,\beta,\gamma)$ satisfying (1) and such that $\sum_{p\in L_0}\Phi(\mathbf{m},\mathbf{n})(p)\not\equiv 0 \pmod 2$. But $(L_1,\alpha,\beta,\gamma)$ has other nonzero characteristic numbers, namely those given by divisors of $\Phi(\mathbf{m},\mathbf{n})$. More precisely, if $\mathbf{m}'=(m_\alpha',m_\beta',m_\gamma')$ and $\mathbf{n}'=(n_\psi')_{\psi\in S'}$, with m_α', m_β', m_γ', n_ψ' equal to 0 or 1, $\sum_{\psi\in S'}n_\psi'>0$, and $m_\alpha'\le m_\alpha$, $m_\beta'\le m_\beta$, $m_\gamma'\le m_\gamma$, $n_\psi'\le n_\psi$ for all $\psi\in S'$, then $\sum_{p\in L_0}\Phi(\mathbf{m}',\mathbf{n}')(p)\not\equiv 0 \pmod 2$. By induction on the number of nontrivial factors of $\Phi(\mathbf{m},\mathbf{n})$ we may assume that, for each such $(\mathbf{m}',\mathbf{n}')$ with $(\mathbf{m}',\mathbf{n}')\neq(\mathbf{m},\mathbf{n})$, there exists $(L_1',\alpha',\beta',\gamma')$ satisfying (1) and (2), so by taking the disjoint union of $(L_1,\alpha,\beta,\gamma)$ with all such $(L',\alpha',\beta',\gamma')$ we obtain the desired result. □

3.3. **Step 2. Construction of** (L_2,φ)**.** Suppose we are given $(L_1,\alpha,\beta,\gamma)$ such that $\tilde\Omega\,\alpha=\tilde\Omega\,\beta=\tilde\Omega\,\gamma=0$ and $\alpha\beta,\alpha\gamma,\beta\gamma$, $\alpha\beta\gamma$, and $\mathbf{1}_{L_1}$ are euler. Note that the sets $(L_1,\alpha,\beta,\gamma)$

constructed in step 1 (section 3.2) are all are actually constructed by wedging blocks of the type

$(P, \alpha, \beta, \gamma) \quad =$



where $\alpha' \equiv \alpha''$ (mod 2), $\beta' \equiv \beta''$ (mod 2), and $\gamma' \equiv \gamma''$ (mod 2).

3.3.1. *Cycles and the values of α, β, γ mod 2.* Let (L_2, φ) be such that $\tilde{\Lambda}\,\varphi = 0$, and let $S \subset L_2$ be a 1-dimensional stratum. Suppose that S is in the boundary of exactly k two-dimensional strata $S_1, \dots, S_k$, with the values of φ on the strata equal to $l_1, \dots, l_k$, respectively. Let $p \in S$. Then, since $\tilde{\Lambda}\,\varphi = 0$,

$$\begin{aligned} \alpha(p) &= \varphi(p) = \tfrac{1}{2}\sum l_i, \\ \beta(p) &= \tilde{\Lambda}\,\varphi^2(p) = \varphi^2(p) - \tfrac{1}{2}\sum l_i^2, \\ \gamma(p) &= \tilde{\Lambda}\,\varphi^3(p) = \varphi^3(p) - \tfrac{1}{2}\sum l_i^3, \end{aligned} \tag{3.4}$$

which are formulas analogous to (3.2).

Consider a block $(P, \alpha, \beta, \gamma)$ as pictured above. By the computations of (3.2.1) there exist $l_1, \dots, l_k$ such that (3.4) gives (α, β, γ) modulo 2. (The value $(\alpha, \beta, \gamma) \equiv (1, 0, 0) \pmod 2$ is given by $(l_1, l_2) = (1, 1)$; $(\alpha, \beta, \gamma) \equiv (0, 1, 0) \pmod 2$ is given by $(l_1, l_2) = (1, -1)$; $(\alpha, \beta, \gamma) \equiv (1, 1, 1) \pmod 2$ is given by $(l_1, l_2) = (2, 0)$.) Define (L_2, φ) to be the disjoint union of k two-dimensional discs D_i, $i = 1, \dots, k$, identified along their boundaries, and let φ be equal to l_i on the interior of each disc D_i and equal to $\frac{1}{2}\sum l_i$ on their common boundary. Then (L_2, φ) satisfies some of the desired properties for $(P, \alpha, \beta, \gamma)$; namely, $\tilde{\Lambda}\,\varphi = 0$ and $\alpha \equiv \varphi|_{L_1}$, $\beta \equiv \tilde{\Lambda}\,\varphi^2$, $\gamma \equiv \tilde{\Lambda}\,\varphi^3 \pmod 2$ on open 1-dimensional strata.

We generalize this construction as follows.

Proposition 3.4. *Let $(L_1, \alpha, \beta, \gamma)$ be such that $\tilde{\Omega}\,\alpha = \tilde{\Omega}\,\beta = \tilde{\Omega}\,\gamma = 0$ and $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\alpha\beta\gamma$, and $\mathbf{1}_{L_1}$ are euler. There exists (L_2, φ) such that*

1. $\tilde{\Lambda}\,\varphi = 0$,
2. $\alpha \equiv \varphi|_{L_1}$, $\beta \equiv \tilde{\Lambda}\,\varphi^2$, $\gamma \equiv \tilde{\Lambda}\,\varphi^3 \pmod 2$ *on open 1-dimensional strata.*

Proof. It is easy to see the hypotheses imply that, for all $(a, b, c) \in (\mathbb{Z}/2)^3$, the set $L(a, b, c) = \{p \in L_1 \mid (\alpha, \beta, \gamma) \equiv (a, b, c) \pmod 2\}$ is a 1-cycle (mod 2). We construct (L_2, φ) by induction on the number of edges (open 1-strata) of L_1 on which (α, β, γ) is nontrivial (mod 2). Suppose $(\alpha, \beta, \gamma) \equiv (a, b, c) \not\equiv (0, 0, 0) \pmod 2$ on some edge of L_1. Since $L(a, b, c)$ is a cycle (mod 2), there is a subset $C(a, b, c)$ of L_1 such that $C(a, b, c)$ is a union of closures of edges of L_1, $C(a, b, c)$ is homeomorphic to a circle, and $(\alpha, \beta, \gamma) \equiv (a, b, c) \pmod 2$ on every edge of $C(a, b, c)$. Suppose (L_2', φ') satisfies (1) and (2) for the set L_1' obtained by removing the edges of $C(a, b, c)$ from L_1. Then we obtain (L_2, φ) from (L_2', φ') by attaching discs along $C(a, b, c)$ just as we did for $(P, \alpha, \beta, \gamma)$ above. $\square$

3.3.2. *Wedges, bubbles, and the values of* α, β, γ *mod* 4. Suppose we are given two 2-dimensional stratified sets (L', φ'), $\tilde\Lambda\,\varphi' = 0$, and (L'', φ''), $\tilde\Lambda\,\varphi'' = 0$, with distinguished 1-dimensional strata $S' \subset L'$ and $S'' \subset L''$. Identifying S' and S'' we produce a new 2-dimensional set L with distinguished 1-stratum S. (If the boundary of S' in L' is a single point p and the boundary of S'' in L'' is two points p_1 and p_2, then both p_1 and p_2 are identified with p.) The functions $\varphi'|_{L'\setminus\bar S'}$ and $\varphi''|_{L''\setminus\bar S''}$ extend to a constructible function φ on L satisfying $\tilde\Lambda\,\varphi = 0$. The function φ is uniquely defined except at the vertices on the boundary of S. We call (L, φ, S) the *wedge* of (L', φ', S') and (L'', φ'', S''), and we denote it by $(L', \varphi', S') \vee (L'', \varphi'', S'')$. The values of $\alpha = \varphi|_{L_1}$, $\beta = \tilde\Lambda\,\varphi^2$, $\gamma = \tilde\Lambda\,\varphi^3$ on S, are determined by the values of $\alpha' = \varphi'|_{L_1'}$, $\beta' = \tilde\Lambda(\varphi')^2$, $\gamma' = \tilde\Lambda(\varphi')^3$, and $\alpha'' = \varphi''|_{L_1''}$, $\beta'' = \tilde\Lambda(\varphi'')^2$, $\gamma'' = \tilde\Lambda(\varphi'')^3$, on S' and S'', respectively, by

$$\begin{aligned}\alpha &= \alpha' + \alpha'',\\ \beta &= \beta' + \beta'' + 2\alpha'\cdot\alpha'',\\ \gamma &= \gamma' + \gamma'' + 3\alpha'\cdot\alpha''(\alpha' + \alpha'').\end{aligned} \tag{3.5}$$

In particular, as we have noticed already for the analogous situation in section 3.2, (α, β, γ) (mod 2) is additive with respect to the operation of taking the wedge. On the other hand, it is not in general additive if we consider the values mod 4.

Fix $l \in \mathbb{Z}$. Let B_l denote the 2-dimensional sphere stratified by the eastern and western hemispheres, the 0° and 180° meridians, and the north and south poles. Set φ on B_l to be constant and equal to l. The 0° meridian will be the distinguished 1-dimensional stratum S_l of B_l. The triple (B_l, φ, S_l) will be called a *bubble*.

Proposition 3.5. *Let* (L_2, φ), *be such that* $\tilde\Lambda\,\varphi = 0$, *and let* S *be a 1-dimensional stratum of* L_2. *Let* α_0, β_0, γ_0 *denote the values of* φ, $\tilde\Lambda\,\varphi^2$, $\tilde\Lambda\,\varphi^3$ *on* S. *Let* $\alpha_0' \equiv \alpha_0 \pmod 2$, $\beta_0' \equiv \beta_0 \pmod 2$, $\gamma_0' \equiv \gamma_0 \pmod 2$. *Then there exist* $l_1, \dots, l_k$, *such that the wedge of* (L_2, φ, S) *along* S *with the bubbles* $(B_{l_i}, \varphi, S_{l_i})$ *has the values* $(\alpha, \beta, \gamma) \equiv (\alpha_0', \beta_0', \gamma_0') \pmod 4$ *on the distinguished stratum of the wedge.*

Proof. When we wedge the bubbles B_l along their distinguished strata we may get any combination of even values of α, β, γ modulo 4. In particular we have:

$$\begin{aligned} B_1 \vee B_3 &\ \text{ gives } (\alpha, \beta, \gamma) = (0, 2, 0),\\ (\vee_3 B_1) \vee B_2 \vee B_3 &\ \text{ gives } (\alpha, \beta, \gamma) = (0, 0, 2),\\ (\vee_4 B_1) \vee B_2 &\ \text{ gives } (\alpha, \beta, \gamma) = (2, 0, 0).\end{aligned}$$

So the proposition follows from the formulas (3.5). □

Corollary 3.6. *Given* $(L_1, \alpha, \beta, \gamma)$ *such that* $\tilde\Omega\,\alpha = \tilde\Omega\,\beta = \tilde\Omega\,\gamma = 0$ *and* $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\alpha\beta\gamma$, *and* $\mathbf{1}_L$ *are euler, then there exists* (L_2, φ) *satisfying the claim of step 2.*

Proof. Propositions 3.4 and 3.5 guarantee the existence of (L_2, φ) such that

1. $\tilde\Lambda\,\varphi = 0$.
2. $\alpha \equiv \varphi|_{L_1}, \beta \equiv \tilde\Lambda\,\varphi^2, \gamma \equiv \tilde\Lambda\,\varphi^3 \pmod 4$ on open 1-dimensional strata.

We establish 2 (mod 2) at the vertices. First note that we may arbitrarily change the value of φ at the vertices without destroying property 1. Hence we may arbitrarily adjust the values of φ so that 2 holds for α. Then the analogous property for β and γ is satisfied automatically. Indeed, the difference $\psi = \beta - \tilde{\Lambda}\,\varphi^2$ is supported in L_1 and is divisible by 4 on open 1-strata. Therefore the property $\tilde{\Omega}\,\psi = 0$ gives that ψ is divisible by 2 on L_0, as required. A similar argument works for γ.

Finally if we wedge L_2 with a 2-sphere S^2 at vertices $p' \in L_2$ and $p'' \in S^2$, with $\varphi = 1$ on S^2, then we change the integral of φ without changing any of the previous properties. This completes the proof of the corollary and step 2. □

3.4. **Step 3. Construction of L_3.** It suffices to prove the following result.

Proposition 3.7. *Let φ be a constructible function defined on a compact semialgebraic set Y of dimension ≤ 2, and suppose $\Lambda\varphi = 0$. There exists a compact semialgebraic set $\tilde{Y}$ of dimension ≤ 3, such that $Y \subset \tilde{Y}$, $\tilde{Y}$ is euler,* $\operatorname{supp}\tilde{\Omega}\,\mathbf{1}_{\tilde{Y}} \subset Y$, *and* $\varphi \equiv \tilde{\Omega}\,\mathbf{1}_{\tilde{Y}} \pmod{2\mathcal{I}(Y)}$.

Proof. Let T be a triangulation of Y such that φ is constant on open simplices of T. For each simplex Δ of T we construct a set $\tilde{\Delta}$, with $\Delta \subset \tilde{\Delta}$. Then $\tilde{Y}$ will be defined as Y with each $\tilde{\Delta}$ attached along the simplex Δ, for all $\Delta \in T$. In other words, if Z is the disjoint union of all the sets $\tilde{\Delta}$ for $\Delta \in T$, then $\tilde{Y}$ is obtained from the disjoint union of Y and Z by identifying each point $y \in Y$ with its image in $\tilde{\Delta}$, for all $\Delta \in T$ containing y.

Given $\Delta \in T$, let $\varphi(\Delta)$ be the value of φ on the interior of Δ. The construction of $\tilde{\Delta}$ depends on the dimension of Δ.

$\dim\Delta = 0$: Then Δ is a one point space say $\Delta = \{p\}$. Let $m = m(\Delta)$ be a positive integer such that $m \equiv 1 - \varphi(\Delta) \pmod 2$. Then $\tilde{\Delta}$ is defined to be the wedge of m circles at p.

$\dim\Delta = 1$: Let $m = m(\Delta)$ be a positive integer such that $m \equiv \varphi(\Delta) \pmod 4$. Then $\tilde{\Delta}$ is defined to be the union of m segments with their left endowment identified and their right endowment identified; that is, $\tilde{\Delta}$ is homeomorphic to the suspension of m points. We identify Δ with one of the segments.

$\dim\Delta = 2$: Let $m = m(\Delta)$ be a positive integer such that $m \equiv 1 - \varphi(\Delta) \pmod 8$. Let R_m denote the wedge of m circles and let $x_0 \in R_m$ denote the base point of the wedge. Then $\tilde{\Delta}$ is defined to be $\Delta \times R_m/\sim$, where we collapse $\{p\} \times R_m$ for each $p \in \partial\Delta$; that is, $(p, x) \sim (q, y)$ if and only if $p = q \in \partial\Delta$ or $p = q \in \operatorname{int}(\Delta)$ and $x = y$. Δ is identified with $\Delta \times \{x_0\}$.

Let $\tilde{Y}$ be the union of Y and the sets $\tilde{\Delta}$ attached along the simplices Δ. Then, since $\tilde{Y} \setminus Y$ is a union of open 1-cells and open 3-cells, $\operatorname{supp}\tilde{\Omega}\,\mathbf{1}_{\tilde{Y}} \subset Y$. Let $\Delta^2 \in T$ be a simplex of dimension 2, and let $p \in \operatorname{int}(\Delta^2)$. Then

$$\tilde{\Omega}\,\mathbf{1}_{\tilde{Y}}(p) = 1 - m(\Delta^2) \equiv \varphi(p) \pmod 8, \tag{3.6}$$

as required.

Let $\Delta^1 \in T$ be a simplex of dimension 1, and let $p \in \operatorname{int}(\Delta^1)$. Let W be a normal slice to Δ^1 in $\tilde{Y}$. (In other words, $W = H \cap \tilde{Y}$, where Y is realized as a semialgebraic set in $\mathbb{R}^n$ and H is an $(n-1)$-plane in $\mathbb{R}^n$ which meets Δ^1 transversally at an interior point of Δ^1.) Let $\psi = \varphi - \tilde{\Omega}\,\mathbf{1}_{\tilde{Y}}$. Then $\Lambda\psi = 0$, which gives, by the slice property (section 1.3(d)) $\Omega(\psi|_H) = 0$; that is,

$$2\psi(p) - \Lambda(\psi|_H)(p) = 0.$$

By (3.6), $\Lambda(\psi|_H)(p) \equiv 0 \pmod 8$, and consequently $\psi(p) \equiv 0 \pmod 4$ as required.

Finally, let p be a vertex of X. Let L, resp. $\tilde{L}$, denote the link of Y, resp. $\tilde{Y}$, at p. The intersections of Y and $\tilde{Y}$ with a small sphere centered at p induce cell structures on L and $\tilde{L}$. Denote by C^0 and C^1, the cells of L of dimension 0 and 1, respectively. Let $\hat{\varphi} = \varphi|_L$. For a cell C we denote by $\hat{\varphi}(C)$ the value of $\hat{\varphi}$ on the interior of C. By the slice property (section 1.3(d)), $\Omega\hat{\varphi} = 0$; that is, for each 0-cell C^0 in L, $\hat{\varphi}(C^0) = \frac{1}{2}\sum_{C^1 \supset C^0} \hat{\varphi}(C^1)$. Consequently

$$\sum_{C^0 \subset L} \hat{\varphi}(C^0) = \sum_{C^1 \subset L} \hat{\varphi}(C^1). \tag{3.7}$$

By construction,

$$\begin{aligned} \#\{\text{ 0-cells in } \tilde{L} \} &\equiv 2(1-\varphi(p)) + \sum_{C^0 \subset L} \hat{\varphi}(C^0) \pmod 4 \\ \#\{\text{ 1-cells in } \tilde{L} \} &\equiv \#\{\text{ 1-cells in } L \} \pmod 4 \\ \#\{\text{ 2-cells in } \tilde{L} \} &\equiv \sum_{C^1 \subset L} (1-\hat{\varphi}(C^1)) \pmod 4 \end{aligned} \tag{3.8}$$

By (3.7) and (3.8), $\chi(\tilde{L}) \equiv 2(1-\varphi(p)) \pmod 4$, so

$$\tilde{\Omega}\,\mathbf{1}_{\tilde{Y}}(p) = 1 - \tfrac{1}{2}\chi(\tilde{L}) \equiv \varphi(p) \pmod 2,$$

as required. □

4. Arithmetic operators on algebraically constructible functions

4.1. **Polynomial operators with rational coefficients.** Since the set of algebraically constructible functions $\mathrm{A}(X)$ is a ring, each polynomial P with integer coefficients gives an operator $\varphi \to P(\varphi)$ on $\mathrm{A}(X)$. We note that some other polynomials with rational coefficients have the same property.

Lemma 4.1. *If φ is an algebraically constructible function on X, then $\frac{1}{2}(\varphi^4 - \varphi^2)$ is also algebraically constructible.*

Proof. By (1.2) we may suppose that $\varphi = \sum_{i=1}^s \operatorname{sgn} g_i$, where g_i are polynomials. Then

$$\begin{aligned} \varphi^2 &= \sum (\operatorname{sgn} g_i)^2 + 2\sum_{i<j} \operatorname{sgn}(g_i g_j), \\ \varphi^4 &= \sum (\operatorname{sgn} g_i)^4 + 2\sum_{i<j} \operatorname{sgn}(g_i g_j)^2 \\ &\quad + 4\big(\sum (\operatorname{sgn} g_i)^2\big)\big(\sum_{i<j} \operatorname{sgn}(g_i g_j)\big) + 4\big(\sum_{i<j} \operatorname{sgn}(g_i g_j)\big)^2, \end{aligned}$$

and hence the result. □

The existence of such operators on algebraically constructible functions gives even more local conditions on the topology of real algebraic sets. We will discuss these conditions for the sets of dimension 4 in the following subsections. But first we classify polynomials which give operations on $\mathrm{A}(X)$ and show, in particular, that modulo 8 the ring of all such polynomials

is generated by $\mathbb{Z}[t]$ and $P(t) = \frac{1}{2}(t^4 - t^2)$. First we note that every such polynomial takes integer values on integers and hence can be uniquely written as a sum

$$P = \sum_{p \geq 0} n_p f_p,$$

where $n_p \in \mathbb{Z}$, and $f_p(t)$ are the binomial polynomials:

$$f_p(t) = \frac{t(t-1)\cdots(t-p+1)}{p!}, \quad p = 1, 2, \dots, \quad f_0(t) = 1. \tag{4.1}$$

Theorem 4.2. *Let $P \in \mathbb{Q}[t]$. The following conditions are equivalent:*

(i) *For every real algebraic set X and every algebraically constructible function φ on X, $P(\varphi)$ is algebraically constructible;*

(ii) *P preserves finite formal sums of signs; that is, for all s, all the coefficients a_α, $\alpha = (\alpha_1, \dots, \alpha_s) \in \{0,1,2\}^s$, in the formal expansion*

$$P\big(\sum_{i=1}^{s} \operatorname{sgn} g_i\big) = \sum_\alpha a_\alpha \prod_{i=1}^{s} (\operatorname{sgn} g_i)^{\alpha_i}$$

are integral;

(iii) *$P = \sum_{p \geq 0} n_p f_p$ and each n_p is an integer divisible by $2^{[p/2]}$.*

Proof. Note that (ii) implies (i) easily by [6], [7], *i.e.* by the characterization (1.2) of algebraically constructible functions. On the other hand, (i)$\Longrightarrow$(ii) is not so obvious. We will proceed in a different way and establish (ii)$\Longleftrightarrow$(iii) first.

Denote by $\mathcal{P}$ the set of polynomials satisfying (ii). It is easy to see that $\mathcal{P}$ is a ring invariant by translations, *i.e.* $P(t) \in \mathcal{P}$ if and only if $P(t+1) \in \mathcal{P}$. Define

$$\Delta P(t) = P(t+1) - P(t).$$

Lemma 4.3. *$P \in \mathcal{P}$ if and only if P is integer-valued on integers, $\Delta P \in \mathcal{P}$, and $\frac{1}{2}\Delta(\Delta P) \in \mathcal{P}$.*

Proof. Define

$$\Delta_2 P(t) = P(t+1) - P(t-1). \tag{4.2}$$

Then we may replace the condition $\frac{1}{2}\Delta(\Delta P) \in \mathcal{P}$ in the statement of the lemma by $\frac{1}{2}\Delta_2 P \in \mathcal{P}$. Indeed,

$$\tfrac{1}{2}\Delta_2 P(t+1) = \tfrac{1}{2}(P(t+2) - P(t)) = \tfrac{1}{2}\Delta(\Delta P)(t) + \Delta P(t), \tag{4.3}$$

and hence $\Delta P \in \mathcal{P}$ and $\frac{1}{2}\Delta(\Delta P) \in \mathcal{P}$ if and only if $\Delta P \in \mathcal{P}$ and $\frac{1}{2}\Delta_2 P \in \mathcal{P}$.

Write $P = \sum_{p \geq 0} n_p f_p$, $n_p \in \mathbb{Z}$, and consider the coefficients a_α, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_s) \in \{0,1,2\}^{s+1}$, given by

$$P\big(\sum_{i=0}^{s} \operatorname{sgn} g_i\big) = \sum_\alpha a_\alpha \prod_{i=0}^{s} (\operatorname{sgn} g_i)^{\alpha_i}.$$

Note that a_α are not necessarily integers if $P \notin \mathcal{P}$. Setting $g_0 = 1, -1,$ or 0, we obtain

$$P\big(\pm 1 + \sum_{i=1}^{s} \operatorname{sgn} g_i\big) = \sum_{\alpha'} (a_{0,\alpha'} \pm a_{1,\alpha'} + a_{2,\alpha'}) \prod_{i=1}^{s} (\operatorname{sgn} g_i)^{\alpha_i},$$

$$P\big(\sum_{i=1}^{s} \operatorname{sgn} g_i\big) = \sum_{\alpha'} a_{0,\alpha'} \prod_{i=1}^{s} (\operatorname{sgn} g_i)^{\alpha_i},$$

where $\alpha' = (\alpha_1, \dots, \alpha_s)$. Consequently,

$$\Delta P\big(\sum_{i=1}^{s} \operatorname{sgn} g_i\big) = P\big(1 + \sum_{i=1}^{s} \operatorname{sgn} g_i\big) - P\big(\sum_{i=1}^{s} \operatorname{sgn} g_i\big) = \sum b_{\alpha'} \prod_{i=1}^{s} (\operatorname{sgn} g_i)^{\alpha_i},$$

where

$$b_{\alpha'} = a_{1,\alpha'} + a_{2,\alpha'}. \tag{4.4}$$

Similarly,

$$\tfrac{1}{2}\Delta_2 P\big(\sum_{i=1}^{s} \operatorname{sgn} g_i\big) = \sum_{\alpha'} c_{\alpha'} \prod_{i=1}^{s} (\operatorname{sgn} g_i)^{\alpha_i},$$

where

$$c_{\alpha'} = a_{1,\alpha'}. \tag{4.5}$$

In particular, if all a_α are integers, so are all $b_{\alpha'}$ and $c_{\alpha'}$. This shows that if $P \in \mathcal{P}$ then P is integer-valued on integers, $\Delta P \in \mathcal{P}$, and $\frac{1}{2}\Delta(\Delta P) \in \mathcal{P}$.

Now we show the converse. Suppose that all $b_{\alpha'}$ and $c_{\alpha'}$ are integers. Then all the coefficients $a_{1,\alpha'}$ and $a_{2,\alpha'}$ are also integral. Since a_α are invariant by permutations of $\alpha_0, \alpha_1, \dots, \alpha_s$, this implies that all a_α, except maybe $a_{(0,\dots,0)}$, are integral. But $a_{(0,\dots,0)} = n_0 = P(0)$ is integral by assumption. This completes the proof of the lemma. □

To show (ii)$\Longleftrightarrow$(iii) we note that $\Delta f_p = f_{p-1}$. Clearly f_0 and f_1 are in $\mathcal{P}$. Since $\deg \Delta P < \deg P$, (ii)$\Longleftrightarrow$(iii) follows easily from Lemma 4.3 by induction on the degree of P.

To complete the proof of the theorem we show (i)$\Longrightarrow$(iii). Suppose $P = \sum_{p \ge 0} n_p f_p$ and suppose that there exists at least one p such that $2^{[p/2]} \nmid n_p$. Let p_0 be the smallest p with this property, and let $k = [p_0/2]$. Consider the algebraically constructible function $\varphi_{p_0} = \operatorname{sgn} x_1 + \dots + \operatorname{sgn} x_k + p_0 - k$ on $\mathbb{R}^k$, where $x_1, \dots, x_k$ denote the coordinate functions on $\mathbb{R}^k$. Then φ_{p_0} takes values between 0 and p_0 for p_0 even, and between 1 and p_0 for p_0 odd. Consider the function

$$P(\varphi_{p_0}) = \sum_{p<p_0} n_p f_p(\varphi_{p_0}) + n_{p_0} f_{p_0}(\varphi_{p_0}) + \sum_{p>p_0} n_p f_p(\varphi_{p_0}). \tag{4.6}$$

The first summand is an algebraically constructible function, and the last summand vanishes identically. Note that $f_{p_0}(\varphi_{p_0})$ is the characteristic function of the open first quadrant Q_k of $\mathbb{R}^k$. So (i)$\Longrightarrow$(iii) is a consequence of the following lemma.

Lemma 4.4. *Let Q_k be the open or closed first quadrant in $\mathbb{R}^k$. For $n \in \mathbb{Z}$, the function $n \cdot \mathbf{1}_{Q_k}$ is algebraically constructible if and only if $2^k | n$.*

Proof. Let Q_k denote the closed first quadrant and let Q'_k denote the open first quadrant in $\mathbb{R}^k$. Suppose $\varphi = n \cdot \mathbf{1}_{Q_k}$ is algebraically constructible. Let $\mathbf{1}_{\mathbb{R}^{k-1}}$ be the characteristic function of $\mathbb{R}^{k-1}$, and let $\psi = (\tilde{\Lambda}\,\varphi)\mathbf{1}_{\mathbb{R}^{k-1}}$. Then ψ is algebraically constructible by Theorem 1.1, and $\psi = \frac{n}{2}\mathbf{1}_{Q_{k-1}}$, with $\frac{n}{2} \in \mathbb{Z}$. By induction on k we have $2^{k-1}|\frac{n}{2}$, so $2^k|n$. If $\varphi' = n \cdot \mathbf{1}_{Q'_k}$, and $\psi' = (\tilde{\Lambda}\,\varphi')\mathbf{1}_{\mathbb{R}^{k-1}}$, then $\psi' = (-1)^{k-1}\frac{n}{2}\mathbf{1}_{Q_{k-1}}$, and so again $2^k|n$ by induction.

To prove the converse, let $S \subset \{1, \dots, k\}$, and let $f_S(x_1, \dots, x_k) = (\epsilon_1 x_1^2, \dots, \epsilon_k x_k^2)$, where $\epsilon_i = 1$ for $i \in S$ and $\epsilon_i = 0$ for $i \notin S$. Then

$$2^k \mathbf{1}_{Q_k}(x) = \sum_S \chi(f_S^{-1}(x)),$$
$$2^k \mathbf{1}_{Q'_k}(x) = \sum_S (-1)^{k-|S|}\chi(f_S^{-1}(x)),$$

so these functions are algebraically constructible by (1.1). □

This completes the proof of Theorem 4.2. □

Let $\mathcal{P}$ denote the ring of polynomials in $\mathbb{Q}[t]$ which satisfy one of the equivalent conditions of Theorem 4.2. Let $\mathcal{A}$ denote the ring of polynomials in $\mathbb{Q}[t]$ which take integer values on integers. By condition (iii) of Theorem 4.2, $\mathcal{P}$ is additively generated by $\mathbb{Z}[t]$ and the polynomials in $\mathcal{P}$ whose values are divisible by 4; that is, $\mathcal{P}/(\mathcal{P} \cap 4\mathcal{A})$ is additively generated by $\mathbb{Z}[t]$. This is the reason that polynomial operators on algebraically constructible functions do not give new characteristic numbers for real algebraic set of dimension ≤ 3.

On the other hand, Theorem 4.2 shows that $\mathcal{P}/(\mathcal{P}\cap 8\mathcal{A})$ is additively generated by $2^{[p/2]}f_p(t)$, $p = 0, \dots, 5$. Later on we will use the following equivalent system of additive generators:

$$\begin{aligned} &1,\ t,\ t^2 - t,\ t^3 - t,\ P_4(t) = \tfrac{1}{2}t(t-1)(t-2)(t-3), \\ &P_5(t) = \tfrac{1}{2}t(t-1)(t-2)(t-3)(t-4). \end{aligned} \tag{4.7}$$

Note that, in particular, the polynomial of Lemma 4.1 can be written

$$\tfrac{1}{2}(t^4 - t^2) = \tfrac{1}{2}t(t-1)(t-2)(t-3) + 3t(t-1)^2. \tag{4.8}$$

Theorem 4.2 admits an obvious generalization to the case of polynomials of many variables.

Corollary 4.5. *Let $P \in \mathbb{Q}[t_1, \dots t_s]$. The following conditions are equivalent:*

(i) *For every real algebraic set X and all algebraically constructible functions $\varphi_1, \dots, \varphi_s$ on X, $P(\varphi_1, \dots, \varphi_s)$ is algebraically constructible;*
(ii) *P preserves finite formal sums of signs;*
(iii) *$P = \sum_{p=(p_1,\dots,p_s)} n_p \prod_{i=1}^s f_{p_i}(t_i)$ and each n_p is an integer divisible by $2^{[p_1/2]+\dots+[p_s/2]}$.*

Proof. (ii)$\Longrightarrow$(i) follows again from (1.2). (iii)$\Longrightarrow$(ii) results directly from the analogous implication in Theorem 4.2. The proof of (i)$\Longrightarrow$(iii) is similar to the proof of the corresponding implication in Theorem 4.2. We sketch the proof. Suppose $P = \sum_{p=(p_1,\dots,p_s)} n_p \prod_{i=1}^s f_{p_i}(t_i)$ has integral coefficients, and suppose that there exists at least one p such that $2^{[p_1/2]+\dots+[p_s/2]} \nmid n_p$. Let p_0 be the smallest p (with respect to the total degree or lexicographic order) with this property, and let $k_i = [p_{0i}/2]$. Consider $P(\varphi_{p_{01}}, \dots, \varphi_{p_{0s}})$, where as before $\varphi_{p_{0i}} = \operatorname{sgn} x_1 +$

$\ldots + \operatorname{sgn} x_{k_i} + p_{0i} - k_i$ are algebraically constructible functions on $\mathbb{R}^{k_i}$. Then

$$P(\varphi_{p_{01}}, \ldots, \varphi_{p_{0s}}) = \psi + n_{p_0} \prod_{i=1}^{s} f_{p_{0i}}(\varphi_{p_{0i}}), \tag{4.9}$$

where ψ is algebraically constructible. Now $\prod_{i=1}^{s} f_{p_{0i}}(\varphi_{p_{0i}})$ is the characteristic function of the open first quadrant in $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_s}$ and hence, by Lemma 4.4, the function (4.9) is not algebraically constructible. $\square$

4.2. **Construction of invariants.** Using Theorem 4.2 we now define new depth k euler conditions and characteristic numbers. Since our procedure goes along the lines of section 2, we just outline the construction and emphasize the new features.

4.2.1. *$\mathcal{P}$-rings and $\mathcal{P}$-euler spaces.* Let $\mathcal{P} \subset \mathbb{Q}[t]$ be the set of polynomials described in Theorem 4.2. Given a constructible function φ, we let $\mathcal{P}(\varphi) = \{P(\varphi) \,|\, P \in \mathcal{P}\}$. More generally, if Φ is a set of constructible functions, we denote by $\mathcal{P}(\Phi)$ the ring generated by $\mathcal{P}(\varphi)$ for $\varphi \in \Phi$, *i.e.* the smallest ring containing Φ and closed under the operators given by $P \in \mathcal{P}$. We call $\mathcal{P}(\Phi)$ the *$\mathcal{P}$-ring* generated by Φ.

Suppose now that the semialgebraic set X is homeomorphic to an algebraic set. Then by Theorems 1.1 and 4.2 all functions constructed from $\mathbf{1}_X$ by means of the arithmetic operations $+$, $-$, $*$, and the operators $\tilde{\Lambda}$ and $\mathcal{P}$, are integer-valued.

We say that the constructible function φ on X is *$\mathcal{P}$-euler* if all functions obtained from φ by using the operations $+$, $-$, $*$, $\tilde{\Lambda}$ and $\mathcal{P}$ are integer-valued. The semialgebraic set X is *$\mathcal{P}$-euler* if $\mathbf{1}_X$ is $\mathcal{P}$-euler. Thus a necessary condition for a semialgebraic set X to be homeomorphic to an algebraic set is that X is $\mathcal{P}$-euler.

4.2.2. *A general construction of depth k $\mathcal{P}$-euler invariants.* In order to produce a list of $\mathbb{Z}/2$-valued obstructions the vanishing of which are necessary and sufficient for X to be $\mathcal{P}$-euler, we follow *verbatim* subsections 2.1 and 2.2, just replacing the word "ring" by "$\mathcal{P}$-ring". Thus let $X \subset \mathbb{R}^n$ be a semialgebraic set, with $d = \dim X$. We denote by $\widetilde{\mathcal{P}}(X)$ the ring all functions obtained from $\mathbf{1}_X$ by using the operations $+$, $-$, $*$, $\tilde{\Lambda}$ and $\mathcal{P}$. We define a sequence of subrings of $\widetilde{\mathcal{P}}(X)$,

$$\widetilde{\mathcal{P}}_0(X) \subset \widetilde{\mathcal{P}}_1(X) \subset \widetilde{\mathcal{P}}_2(X) \subset \cdots,$$

where $\widetilde{\mathcal{P}}_0(X)$ is the ring generated by $\mathbf{1}_X$, and for $k \geq 0$, $\widetilde{\mathcal{P}}_{k+1}(X)$ is the $\mathcal{P}$-ring generated by $\widetilde{\mathcal{P}}_k(X)$ and $\{\tilde{\Lambda}\,\varphi \mid \varphi \in \widetilde{\mathcal{P}}_k(X)\}$.

Suppose that $\widetilde{\mathcal{P}}_k(X)$ is integer-valued, and suppose we have subsets $G_0, \ldots, G_{k-1}$ of $\widetilde{\mathcal{P}}_k(X)$ such that for $j = 0, \ldots, k$, the set $G_0 \cup \cdots \cup G_j$ additively generates the ring $\widetilde{\mathcal{P}}_j(X)/(\mathcal{I}(X) \cap \widetilde{\mathcal{P}}_j(X))$, and for all $\varphi \in G_j$, $\operatorname{supp} \varphi \subset X_{d-j}$. We call the conditions that $\varphi \in G_k$ are euler the *depth $k+1$ $\mathcal{P}$-euler conditions* for X.

The compact semialgebraic set L is the link of a point in a $\mathcal{P}$-euler space X if and only if the following conditions are satisfied:

(A) L is $\mathcal{P}$-euler,

(B) For all $\varphi \in \widetilde{\mathcal{P}}(L)$, $\int \varphi \, d\chi$ is even.

We define the *characteristic numbers of depth k* by the same construction as in section 2.2.

4.2.3. *Invariants in dimension 4.* Let L be a compact semialgebraic set of dimension ≤ 3. We work out the conditions (A) and (B) for L to be a link of a $\mathcal{P}$-euler space.

(A) *L is $\mathcal{P}$-euler.* It is easy to see that

$$\widetilde{\mathcal{P}}(L)/(\mathcal{I}(L)\cap\widetilde{\mathcal{P}}(L)) = \tilde{\Lambda}(L)/(\mathcal{I}(L)\cap\tilde{\Lambda}(L)),$$

so L is $\mathcal{P}$-euler if and only of L is completely euler. Indeed $\widetilde{\mathcal{P}}_1(L)/(\mathcal{I}(L)\cap\widetilde{\mathcal{P}}(L))$ is generated additively by $\mathbf{1}_L$, φ, φ^2, φ^3, since $2^{[p/2]}f_p(\varphi)$, $p\geq 4$, are divisible by 4 and therefore in $\mathcal{I}(L)$. Similarly $\widetilde{\mathcal{P}}_2(L)/(\mathcal{I}(L)\cap\widetilde{\mathcal{P}}_2(L))$ is additively generated by $\mathbf{1}_L$, φ, φ^2, φ^3, and the products $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\alpha\beta\gamma$, where $\beta=\tilde{\Lambda}(\varphi^2)$ and $\gamma=\tilde{\Lambda}(\varphi^3)$.

(B)*For all $\varphi\in\widetilde{\mathcal{P}}(L)$, $\int\varphi\,d\chi$ is even.* We compute $\widetilde{\mathcal{P}}_3(L)/(2\mathcal{I}(L)\cap\widetilde{\mathcal{P}}_3(L))$. By Theorem 4.2 (see also (4.7)), the ring $\widetilde{\mathcal{P}}_1(L)/(2\mathcal{I}(L)\cap\widetilde{\mathcal{P}}_1(L))$ is additively generated by

$$\begin{aligned}&\mathbf{1}_L,\ \varphi,\ \varphi_2=(\varphi^2-\varphi),\ \varphi_3=(\varphi^3-\varphi),\\ &\varphi_4=\tfrac{1}{2}\varphi(\varphi-1)(\varphi-2)(\varphi-3),\ \varphi_5=\varphi_4(\varphi-4).\end{aligned}\tag{4.10}$$

Recall that $\varphi_2\equiv\varphi_3\equiv 0 \pmod 2$ and $\varphi_4\equiv\varphi_5\equiv 0 \pmod 4$. So there are no new characteristic numbers of depth one.

Now consider

$$\beta=\tilde{\Lambda}(\varphi_2),\ \gamma=\tilde{\Lambda}(\varphi_3),\ \delta=\tilde{\Lambda}(\varphi_4),\ \epsilon=\tilde{\Lambda}(\varphi_5).\tag{4.11}$$

(This coincides with our previous notation: $\beta=\tilde{\Lambda}(\varphi^2)$ and $\gamma=\tilde{\Lambda}(\varphi^3)$, since $\tilde{\Lambda}\,\varphi=0$.) As before, let $\alpha=\varphi|L_1$, $\alpha_1=\alpha$, $\alpha_2=\alpha^2-\alpha$, $\alpha_3=\alpha^3-\alpha$, and similarly define β_1, β_2, β_3, γ_1, γ_2, γ_3. Let $\alpha_0=\beta_0=\gamma_0=1$. The ring $\widetilde{\mathcal{P}}_2(L)/(2\mathcal{I}(L)\cap\widetilde{\mathcal{P}}_2(L))$ is generated additively by $\mathbf{1}_L$, φ, φ^2, φ^3, together with representatives of the equivalence classes mod 4 of

$$\psi_{abcde}=\alpha_a\beta_b\gamma_c\delta^d\epsilon^e \quad b+c+d+e>0,\tag{4.12}$$

where $a,b,c=0,1,2,3$, and $d,e=0,1$ (since δ and ϵ are even valued). The functions ψ_{abcde} divisible by 4 do not count, so the complete list of generators comprises $S_1\cup S_2\cup S_3\cup S_4$ of (2.3) and the following sets:

$$\delta,\ \epsilon;\tag{S_0}$$

$$\begin{gathered}\alpha\delta,\ \beta\delta,\ \gamma\delta,\ \alpha\epsilon,\ \beta\epsilon,\ \gamma\epsilon,\\ \alpha\beta\delta,\ \alpha\gamma\delta,\ \beta\gamma\delta,\ \alpha\beta\epsilon,\ \alpha\gamma\epsilon,\ \beta\gamma\epsilon,\ \alpha\beta\gamma\delta,\ \alpha\beta\gamma\epsilon.\end{gathered}\tag{S_5}$$

The characteristic numbers of depth 2 are the euler integrals mod 2 of these functions, and again the only nontrivial ones are $\int\alpha\beta\,d\chi$, $\int\alpha\gamma\,d\chi$, $\int\beta\gamma\,d\chi$, $\int\alpha\beta\gamma\,d\chi$.

Let $S=S_0\cup S_1\cup S_2\cup S_3\cup S_4\cup S_5$, and let $S'=S_2\cup S_3\cup S_4\cup S_5$. The ring $\tilde{\Lambda}_3(L)/(2\mathcal{I}(L)\cap\tilde{\Lambda}_3(L))$ is generated additively by $\mathbf{1}_L$, φ, φ^2, φ^3, and S, together with the equivalence classes mod 2 of the functions

$$\Phi(\mathbf{m},\mathbf{n})=\alpha^{m_\alpha}\beta^{m_\beta}\gamma^{m_\gamma}\prod_{\psi\in S'}(\tilde{\Omega}(\psi))^{n_\psi},\tag{4.13}$$

where the exponents $m_\alpha, m_\beta, m_\gamma, n_\psi$ equal 0 or 1, and $\sum_{\psi\in S'}n_\psi>0$.

Remark 4.6. Neither δ, ϵ nor $\tilde{\Omega}(\delta)$, $\tilde{\Omega}(\epsilon)$ appear as factors in (4.13), since δ, ϵ are even valued and $\tilde{\Omega}(\delta)=\tilde{\Omega}(\epsilon)=0$.

The support of $\Phi(\mathbf{m},\mathbf{n})$ is contained in L_0, and hence

$$\int \Phi(\mathbf{m},\mathbf{n})\, d\chi = \sum_{p\in L_0} \Phi(\mathbf{m},\mathbf{n})(p). \tag{4.14}$$

If $\psi \in S_3 \cup S_4 \cup S_5$ then $\int \psi\, d\chi$ is automatically even. This leaves us with $2^3(2^{40}-1)-40 = 2^{43}-48$ new characteristic numbers (4.14) of depth 3. Thus we obtain:

Theorem 4.7. *Let L be a compact semialgebraic set of dimension at most 3. Let $\varphi = \tilde\Omega\, \mathbf{1}_L$, $\beta = \tilde\Lambda\, \varphi^2$, $\gamma = \tilde\Lambda\, \varphi^3$, $\delta = \tilde\Lambda\, \varphi_4$, $\epsilon = \tilde\Lambda\, \varphi_5$. Then L is a link in a $\mathcal{P}$-euler space if and only if the following conditions hold:*

1. *L is euler and $\varphi\beta$, $\varphi\gamma$, $\beta\gamma$, $\varphi\beta\gamma$ are euler;*
2. *L has even euler characteristic and $\varphi\beta$, $\varphi\gamma$, $\beta\gamma$, $\varphi\beta\gamma$ have even euler integral;*
3. *The $2^{43}-48$ characteristic numbers of (4.14)*
$$\mathbf{a}(\mathbf{m},\mathbf{n})(L) = \sum_{p\in L_0} \Phi(\mathbf{m},\mathbf{n})(p),$$
are even.

4.3. **Independence of invariants.** We show that the $2^{43}-43$ characteristic numbers of (2) and (3) of Theorem 4.7 are independent. For each characteristic number we construct a set L, together with a filtration $L_0 \subset L_1 \subset L_2 \subset L_3 = L$ by skeletons of a stratification, with exactly the given characteristic number nonzero. For the characteristic number $\chi(L) \pmod 2$ we can again use $L = S^3 \vee S^3$.

4.3.1. *Step 1.* Given a characteristic number $\mathbf{a}(\mathbf{m},\mathbf{n})$, we construct L_1 and constructible functions α, β, γ, δ, ϵ on L_1 such that:

(a) $\tilde\Omega\,\alpha = \tilde\Omega\,\beta = \tilde\Omega\,\gamma = \tilde\Omega\,\delta = \tilde\Omega\,\epsilon = 0$, $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\alpha\beta\gamma$, and $\mathbf{1}_{L_1}$ are euler, and δ, ϵ are even-valued,

(b) $\sum_{p\in L_0} \Phi(\mathbf{m},\mathbf{n})(p)$ is the only sum as in (4.14) which is nonzero (mod 2).

We use the elementary blocks (α), (β), (γ), $(\alpha\beta), \dots$, of section 3.2, with the understanding that the values of δ and ϵ on these blocks are identically equal to zero. We introduce new blocks for the functions in S_5. (For the functions in S_0 we do not need elementary blocks because of Remark 4.6.) Since $\tilde\Omega\,\delta = \tilde\Omega\,\epsilon = 0$, the values of δ and ϵ at the vertices are determined by their values on open 1-dimensional strata.

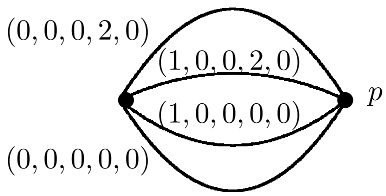
Remark 4.8. There is an important difference between δ and ϵ and the other even-valued functions of S, since the sets $\{p \in L_1 \mid \delta/2 \equiv 1 \pmod 2\}$, $\{p \in L_1 \mid \epsilon/2 \equiv 1 \pmod 2\}$ are automatically 1-cycles (mod 2). This follows immediately from the fact that $\tilde\Omega(\delta/2) = \tilde\Omega(\epsilon/2) = 0$.

At each vertex of our new elementary blocks we consider, in addition to $V_1 \cup V_2 \cup V_3 \cup V_4$, the following 14 new expressions:

$$\begin{gathered}\tilde\Omega(\alpha\delta)(p),\ \tilde\Omega(\beta\delta)(p),\ \tilde\Omega(\gamma\delta)(p),\ \tilde\Omega(\alpha\epsilon)(p),\ \tilde\Omega(\beta\epsilon)(p),\ \tilde\Omega(\gamma\epsilon)(p),\\ \tilde\Omega(\alpha\beta\delta)(p),\ \tilde\Omega(\alpha\gamma\delta)(p),\ \tilde\Omega(\beta\gamma\delta)(p),\ \tilde\Omega(\alpha\beta\epsilon)(p),\ \tilde\Omega(\alpha\gamma\epsilon)(p),\ \tilde\Omega(\beta\gamma\epsilon)(p),\\ \tilde\Omega(\alpha\beta\gamma\delta)(p),\ \tilde\Omega(\alpha\beta\gamma\epsilon)(p).\end{gathered} \tag{V_5}$$

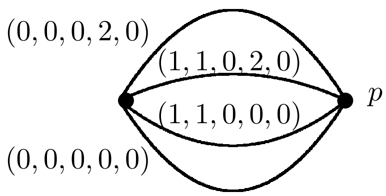
The elementary blocks corresponding to V_5 are defined by simple pictures. We illustrate 3 cases; the remaining 11 elementary blocks are defined by symmetry among the functions α, β, and γ, and by symmetry between δ and ϵ.

$(\alpha\delta)$



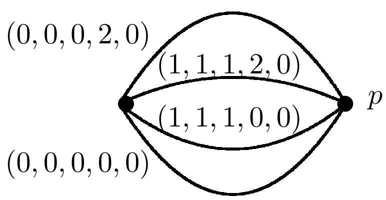
$\tilde{\Omega}(\alpha\delta)(p) \neq 0$

$(\alpha\beta\delta)$



$\tilde{\Omega}(\alpha\beta\delta)(p) \neq 0$

$(\alpha\beta\gamma\delta)$



$\tilde{\Omega}(\alpha\beta\gamma\delta)(p) \neq 0$

Let $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$. For each expression v in V the corresponding elementary block again has the property that v is nonzero at all vertices of the block, and all expressions which follow v in the linear ordering of V vanish at both vertices of the block.

The rest of step 1 goes exactly as in step 1 of section 2, since the expressions of V_5 are additive under wedging.

4.3.2. *Step 2.* Given $(L_1, \alpha, \beta, \gamma, \delta, \epsilon)$ satisfying (a) of step 1, we construct L_2 and a constructible function φ on L_2 such that:

(a) $\tilde{\Lambda}\,\varphi = 0$,
(b) $\alpha \equiv \varphi|_{L_1}$, $\beta \equiv \tilde{\Lambda}\,\varphi_2$, $\gamma \equiv \tilde{\Lambda}\,\varphi_3$, $\delta \equiv \tilde{\Lambda}\,\varphi_4$, $\epsilon \equiv \tilde{\Lambda}\,\varphi_5$ (mod $2\mathcal{I}(L)$),
(c) $\int \varphi\, d\chi$ is even.

Let (L_2, φ) be such that $\tilde{\Lambda}\,\varphi = 0$, and let $S \subset L_2$ be a 1-dimensional stratum. Suppose that S is in the boundary of exactly k two-dimensional strata $S_1, \dots, S_k$, with the values of

φ on the strata equal to $l_1, \dots, l_k$, respectively. Let $p \in S$. Then, since $\tilde{\Lambda}\,\varphi = 0$,

$$\begin{aligned}
\alpha(p) &= \varphi(p) = \tfrac{1}{2}\sum l_i, \\
\beta(p) &= \tilde{\Lambda}\,\varphi_2(p) = \varphi_2(p) - \tfrac{1}{2}\sum (l_i^2 - l_i), \\
\gamma(p) &= \tilde{\Lambda}\,\varphi_3(p) = \varphi_3(p) - \tfrac{1}{2}\sum (l_i^3 - l_i), \\
\delta(p) &= \varphi_4(p) - \frac{1}{4}\sum l_i(l_i-1)(l_i-2)(l_i-3), \\
\epsilon(p) &= \varphi_5(p) - \frac{1}{4}\sum l_i(l_i-1)(l_i-2)(l_i-3)(l_i-4).
\end{aligned} \tag{4.15}$$

Lemma 4.9. *Given $(a,b,c,d,e) \in (\mathbb{Z}/2)^5$, there exist $l_1, \dots, l_k$, such that $(\alpha,\beta,\gamma,\delta,\epsilon)$ given by (4.15) satisfy $(\alpha,\beta,\gamma,\frac{1}{2}\delta,\frac{1}{2}\epsilon) \equiv (a,b,c,d,e) \pmod 2$.*

Proof. We have $\varphi_2 \equiv \varphi_3 \equiv \frac{1}{2}\varphi_4 \equiv \frac{1}{2}\varphi_5 \equiv 0 \pmod 2$ and consequently β, γ, $\frac{1}{2}\delta$, $\frac{1}{2}\epsilon \pmod 2$ are additive with respect to the operation of taking the wedge at p. So is α. Let $P_2(t) = t^2 - t$, $P_3(t) = t^3 - t$, $P_4(t) = \frac{1}{2}t(t-1)(t-2)(t-3)$, $P_5(t) = P_4(t)(t-4)$. Then

$$\begin{aligned}
&(\tfrac{1}{2}P_2(1), \tfrac{1}{2}P_3(1), \tfrac{1}{2}P_4(1), \tfrac{1}{2}P_5(1)) \equiv (0,0,0,0) \pmod 2, \\
&(\tfrac{1}{2}P_2(2), \tfrac{1}{2}P_3(2), \tfrac{1}{2}P_4(2), \tfrac{1}{2}P_5(2)) \equiv (1,1,0,0) \pmod 2, \\
&(\tfrac{1}{2}P_2(3), \tfrac{1}{2}P_3(3), \tfrac{1}{2}P_4(3), \tfrac{1}{2}P_5(3)) \equiv (1,0,0,0) \pmod 2, \\
&(\tfrac{1}{2}P_2(4), \tfrac{1}{2}P_3(4), \tfrac{1}{2}P_4(4), \tfrac{1}{2}P_5(4)) \equiv (0,0,1,0) \pmod 2, \\
&(\tfrac{1}{2}P_2(5), \tfrac{1}{2}P_3(5), \tfrac{1}{2}P_4(5), \tfrac{1}{2}P_5(5)) \equiv (0,0,1,1) \pmod 2.
\end{aligned} \tag{4.16}$$

So the lemma follows easily from the additivity property. □

Proposition 4.10. *Let $(L_1, \alpha, \beta, \gamma, \delta, \epsilon)$ be such that $\tilde{\Omega}\,\alpha = \tilde{\Omega}\,\beta = \tilde{\Omega}\,\gamma = 0$, $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\alpha\beta\gamma$, $\mathbf{1}_{L_1}$ are euler, and δ, ϵ are even-valued. There exists (L_2, φ) such that*

(a) $\tilde{\Lambda}\,\varphi = 0$,
(b) $\alpha \equiv \varphi|_{L_1}$, $\beta \equiv \tilde{\Lambda}\,\varphi^2$, $\gamma \equiv \tilde{\Lambda}\,\varphi^3$, $\frac{1}{2}\delta \equiv \frac{1}{2}\tilde{\Lambda}\,\varphi_4$, $\frac{1}{2}\epsilon \equiv \frac{1}{2}\tilde{\Lambda}\,\varphi_5 \pmod 2$ *on open 1-dimensional strata.*

Proof. By Proposition 3.4 there exists (L_2', φ) such that $\tilde{\Lambda}\,\varphi = 0$, and $\alpha \equiv \varphi|_{L_1}$, $\beta \equiv \tilde{\Lambda}\,\varphi^2$, $\gamma \equiv \tilde{\Lambda}\,\varphi^3 \pmod 2$ on open 1-dimensional strata. Moreover, the proofs of Proposition 3.4 and Lemma 4.9 actually show the existence of such (L_2', φ) with the additional properties that $\tilde{\Lambda}\,\varphi_4 = \tilde{\Lambda}\,\varphi_5 = 0$, for (L_2', φ) can be obtained from L_1 by attaching discs D_i with $\varphi|\operatorname{int} D_i$ equal to 1, 2, or 3 (by (4.16)).

Now by Remark 4.8 the sets $\{p \in L_1 \,|\, \delta/2 \equiv 1 \pmod 2\}$, $\{p \in L_1 \,|\, \epsilon/2 \equiv 1 \pmod 2\}$ are 1-cycles (mod 2). So by attaching further discs D_i to L_1 as in Proposition 3.4, with $\varphi|\operatorname{int} D_i$ equal to 4 or 5 (by (4.16)), we achieve the desired result. □

Now step 2 is completed just as before. By wedging along 1-strata with bubbles $(B_{l_i}, \varphi, S_{l_i})$ as in Proposition 3.5, we can adjust the values of $\varphi|_{L_1}$, $\tilde{\Lambda}\,\varphi^2$, $\tilde{\Lambda}\,\varphi^3 \pmod 4$ to agree with α, β, γ, respectively, without changing $\tilde{\Lambda}\,\varphi_4$ or $\tilde{\Lambda}\,\varphi_5$. As a result we have (L_2, φ) such that $\tilde{\Lambda}\,\varphi = 0$, $\alpha \equiv \varphi|_{L_1}$, $\beta \equiv \tilde{\Lambda}\,\varphi_2$, $\gamma \equiv \tilde{\Lambda}\,\varphi_3$, $\delta \equiv \tilde{\Lambda}\,\varphi_4$, $\epsilon \equiv \tilde{\Lambda}\,\varphi_5 \pmod 4$ on open 1-dimensional strata. Then, as in the proof of Corollary 3.6, we adjust the values of φ on L_0 so that $\alpha \equiv \varphi \pmod 2$

on L_0, and it follows that the congruences for β, γ, δ, and ϵ hold mod 2 on L_0. Finally we can achieve that $\int \varphi \, d\chi \equiv 0 \pmod 2$ by wedging L_2 with a 2-sphere at a point.

4.3.3. *Step 3.* Given (L_2, φ) satisfying (a) of step 2, we construct L such that L is euler and $\varphi \equiv \tilde{\Omega}\, \mathbf{1}_L \pmod{2\mathcal{I}(L)}$. This is identical to our previous construction (section 3.4).

This completes the construction of examples, and hence the proof of the independence of the invariants (2) and (3) of Theorem 4.7.

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